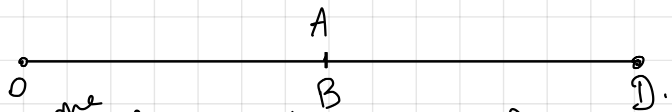
February 25th. 2026.

Course #5.

Midterm. Wednesday 4 p.m.
1 hour.

Hotelling's model.

(1) competing on location



only one Nash equilibrium. They concentrate.

(2) Competition on prices.

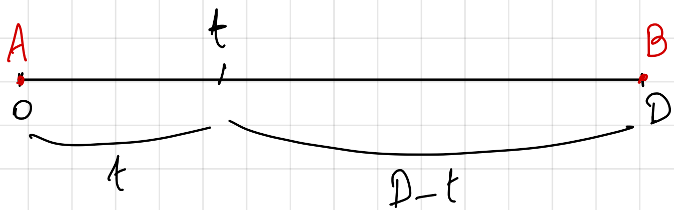
- we fix firms' location
  - Firm A is at 0
  - Firm B is at D.
- They must choose p_A and p_B.

, The customers attach some value V to the good.

If a customer's location is at t with $0 \leq t \leq D$,

her utility from buying from

- firm A : $V - p_A - ct$
- firm B : $V - p_B - c(D-t)$



If prices are too high, consumers don't buy.

How to determine the demand for each firm?

We have to find t^* such that

$$V - p_A - ct^* = V - p_B - c(D - t^*)$$

$$\Leftrightarrow \quad p_B - p_A = -cD + ct^* + ct^*$$

$$\Leftrightarrow \frac{cD + p_B - p_A}{2c} = t^*.$$

$$t^* = \frac{D}{2} + \frac{p_B - p_A}{2c}$$

customers $\sim U[0;D]$
from 0 to t^* we have t^* consumers.

Demand to firm A = all customers from 0 to t^*.

" " " B = " " " t^* to D.

Suppose marginal cost of production is 0.

$$\pi_A = p_A \times t^* = p_A \times \left(\frac{D}{2} + \frac{p_B - p_A}{2c}\right).$$

$$\pi_B = p_B \times (D - t^*) = p_B \times \left(\frac{D}{2} - \frac{p_B - p_A}{2c}\right).$$

$$D - \left[\frac{D}{2} + \frac{p_B - p_A}{2c}\right]$$

$$= D - \frac{D}{2} - \frac{p_B - p_A}{2c}$$

Next: maximize the profit functions.

$$\frac{d\pi_A}{dp_A} = \frac{D}{2} + \frac{p_B}{2c} - \frac{p_A}{c} = 0$$

$$\pi_A = p_A \frac{D}{2} + p_A \frac{p_B}{2c} - \left(p_A \frac{p_A}{2c}\right) \Rightarrow = \frac{p_A^2}{2c}$$

$$\frac{d\pi_A}{dp_A} = \frac{D}{2} + \frac{p_B}{2c} - \frac{1}{\cancel{2}c} \times \cancel{2} p_A = 0 \iff p_A = \frac{cD + p_B}{2}$$

Best response function for firm A.

$$\iff \frac{cD}{2} + \frac{\cancel{c} p_B}{2\cancel{c}} = p_A$$

$$\frac{d\pi_B}{dp_B} = 0 \iff p_B = \frac{cD + p_A}{2}$$

Best response function for firm B.

$$\pi_B = p_B \times \frac{D}{2} + p_B \times \frac{p_A}{2c} - p_B \times \frac{p_B}{2c}$$

$$\frac{d\pi_B}{dp_B} = \frac{D}{2} + \frac{p_A}{2c} - \frac{p_B}{c} = 0 \iff p_B = \frac{cD + p_A}{2}.$$

$$\text{NE: } \begin{cases} p_A = \dfrac{cD + p_B}{2} \\ p_B = \dfrac{cD + p_A}{2} \end{cases} \iff p_A^* = p_B^* = cD.$$

<u>conclusions</u> :

- higher c → higher prices
- higher D → higher prices.

$$f(x) = kx$$

$$f'(x) = \frac{df(x)}{dx} = k$$

$$[u(x) \times v(x)]' = u'v + v'u$$

3 what if firms can decide location and prices

<u>two opposing forces</u> :

- Attraction force : close to each other to steal the customers.
- Repulsion force : they monopoly power in their neighborhood.

which force is stronger ? No definite answer.

CHAPTER 4. Mixed strategy equilibrium

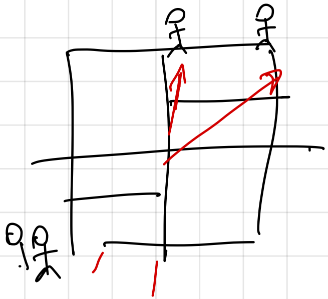
matching pennies

		pl 2: H	pl 2: T
pl 1	H	$1,-1$	$-1,1$
	T	$-1,1$	$1,-1$

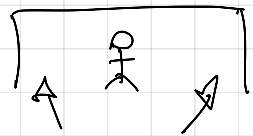
- no Nash equilibrium in pure strategy.
- we introduce the possibility to randomize over the actions.

$\rightarrow$ a strategy consists in choosing a probability distribution over the actions.

Wooders.



Soccer.



Zinedine Zidane.

So far: ordinal utility function

- complete : always order the outcomes
- transitive : $A \succsim B \succsim C$ then $A \succsim C$.

$A \succsim B$ and $B \succsim C$

- do you prefer
  - obtaining A with probability 0.5 and C with probability 0.5
  - or obtaining B for sure.

$\rightarrow$ we need 2 more assumptions to define CARDINAL utility function.

- Continuity $\boxed{A \succsim B \succsim C}$ then $\exists\, p \in [0;1]$ s.t. $pA + (1-p)C \sim B$.
- Independence $A \succsim B$, then for any

outcome D and $p \in [0;1]$

$$pA + (1-p)D \succsim pB + (1-p)D.$$

With these 4 assumptions : Lotteries

$$L = pA + (1-p)B \quad \text{and} \quad M = pC + (1-p)D$$

If $\underset{\text{preferences.}}{L \succsim M} \iff \underbrace{pu(A) + (1-p)u(B)}_{\text{real number}} \geq \underbrace{pu(C) + (1-p)u(D)}_{\text{real number.}}$

expected payoffs.

Back to matching pennies

player 1's set of actions : $A_1 = \{H, T\}$.

mixed strategy : $\alpha_1 : (p, (1-p))$

example : $\alpha_1 = (\frac{1}{2}; \frac{1}{2})$ or $\alpha_1 = (\frac{1}{3}, \frac{2}{3})$

... $\alpha_1 = (1, 0)$.

α_i is a probability distribution defined over A_i.

$\alpha_1 : (p, 1-p)$

$\alpha_2 : (q, 1-q)$

pl 2

		H q	T $(1-q)$
pl 1	H p	I	II
	T $1-p$	III	IV

What is the probability to observe outcome (H, H) I is $p \times q$

II is $p \times (1-q)$

III is $(1-p)\, q$

IV is $(1-p)\times(1-q)$

Player 1. gains 1€ in I and IV.

with probility $pq + (1-p)(1-q)$.

$= pq + 1 - q - p + pq$

$= (1-q) + p(2q-1)$

Looses 1€ in II and III

with proloability : $p(1-q) + (1-p)q$

$= q + p(1-2q)$.

	Player 2	$q=0$	$q=1/3$	$q=\frac{1}{2}$	$q=\frac{2}{3}$	$q=1$
Gain 1€ player 1		$1-p$	$\frac{2}{3}-\frac{p}{3}$	$\frac{1}{2}$	$\frac{1}{3}+\frac{p}{3}$	p

when q is low $(q<1/2)$, then the probability to gain 1€

is decreasing in p. $\longrightarrow$ choose $p=0$. pl 1 plays T for sure

when q is high ($q > \frac{1}{2}$), the probability to gain 1€

is increasing in p $\longrightarrow$ choose $p=1$. player 1 plays H

when q is $= 1/2$ $\longrightarrow$ choose any p.

Conclusion: we know that if player 2 chooses q low (she play T most of the time), then player 1 chooses $p=0$ or she plays T with certainty. She does not randomize. Then player 2 is better off playing H (i.e. $q=1$). This induces player 1 to play H instead of T... etc.

$$\boxed{\text{No Nash equilibrium if } q \neq 1/2.}$$

Feb 26th.
Louise #6

4.1.3. Expected payoffs.

$\succsim$ preferences. • rational → complete, transitive

• continuity $A \succsim B \succsim C$ then $\exists\, p \in [0;1]$ s.t. $pA + (1-p)C \sim B$.

• Independent

if $A \succsim B$ then for any D and any $p \in [0;1]$: $pA + (1-p)D \succsim pB + (1-p)D$.

<u>Theorem:</u> If a preference relation is complete, transitive, continuous and independent, then there exists a cardinal payoff function u such that

$$L = pA + (1-p)B \quad \text{and} \quad M = pC + (1-p)D$$

$$L \succsim M \iff pu(A) + (1-p)u(B) \geq pu(C) + (1-p)u(D)$$

von Neumann - Morgenstern preferences

guarantee the existence of u, a Bernoulli payoff function, a cardinal payoff function whose expected values represent the VNM preferences.

"Preferences regarding lotteries over outcomes may be represented by the expected value of a payoff function over deterministic outcomes".

Example: Let P and Q two lotteries over a, b, c three deterministic outcomes.

$P = (p_1, p_2, p_3)$ with $p_i \geq 0$ $i = 1, 2, 3$
and $\sum p_i = 1$

$Q = (q_1, q_2, q_3) \quad q_i \geq 0 \quad i = 1,2,3$

$\sum q_i = 1$

If $P \succ Q$ then

$$\underbrace{p_1 u(a) + p_2 u(b) + p_3 u(c)}_{\text{real number}} \overset{\text{greater}}{>} \underbrace{q_1 u(a) + q_2 u(b) + q_3 u(c)}_{\text{real number}}$$

Attitude to risk $\quad a \succ b \succ c$

1) risk averse $\quad \underbrace{p\,u(a) + (1-p)\,u(c)}_{\text{risky situation}} \overset{\text{smaller}}{<} \underbrace{u(b)}_{\text{sure outcome}}$ even if

p is large.

$u(a) = 10 \quad u(b) = 9 \quad u(c) = 0$

let $p = \frac{3}{4} \qquad \frac{3}{4} \times 10 + \frac{1}{4} \times 0 = 7.5 < 9.$

2) <u>risk lover</u> $\quad p\,u(a) + (1-p)\,u(c) > u(b)$ even if p is small.

example : $\quad u(a) = 10 \qquad u(b) = 1 \qquad u(c) = 0$

let $p = \frac{1}{4}$ $\qquad \frac{1}{4} \times 10 + \frac{3}{4} \times 0 = 2.5 > 1.$

3) risk neutral $\qquad$ a, b and c are monetary amounts

$$100\,€ \sim \frac{1}{2}(200\,€) + \frac{1}{2}(0\,€).$$

or $\quad 100\,€ \sim \frac{1}{2}(400\,€) + \frac{1}{2}(-200\,€)$

or $\quad 100\,€ \sim \frac{9}{10}(0\,€) + \frac{1}{10}(1000\,€).$

When we talk about risk aversion or lover, then the preferences are not expected monetary amount. They are expected PAYOFFS

≠ money.

4.2. Strategic games with vNM preferences.

- set of players
- set of actions.
- preferences regarding lotteries over actions profiles

before: values of payoffs that represent preferences over deterministic outcomes.

Now: values of Bernoulli function whose represent preferences over lotteries.

Ex:

pl 1 \ pl 2	Q	F
Q	2,2	0,3
F	3,0	1,1

pl 1 \ pl 2	Q	F
Q	3,3	0,4
F	4,0	1,1

these two games are equivalent when preferences are ordinal.

In the left game, player 1 is indifferent between

$$(Q,Q) \quad \text{and} \quad \frac{1}{2}(F,Q) + \frac{1}{2}(F,F).$$

$$2 \qquad \frac{1}{2} \times 3 + \frac{1}{2} \, 1$$

But not on the right game

$$(Q,Q) \qquad \frac{1}{2}(F,Q) + \frac{1}{2}(F,F)$$

$$3 > \frac{1}{2} \times 4 + \frac{1}{2} \times 1 = 2.5$$

4.3. Mixed strategy Nash equilibrium.

- a mixed strategy of a player is a probability distribution over her set of possible actions.

- a mixed strategy is PURE if degenerated (there is an action with proba = 1 and all other actions have proba $= 0$).

Notation :

- set of players $\{1, 2, \dots, n\}$. $\{1, 2\}$
- set of possible actions A_i for player i $A_1 = \{H, T\} = A_2$
- α_i a probability distribution over A_i $\alpha_1 = (\frac{1}{2}, \frac{1}{2})$

$a_i \in A_i$, $\alpha_i(a_i) = \text{Proba}(i \text{ plays } a_i)$. $\alpha_2 = (\frac{1}{3}, \frac{2}{3})$

$\# A_i = m$, $\sum_{k=1}^{m} \alpha_i(a_k) = 1$. $\# A_1 = 2$

$\alpha_1(H) + \alpha_1(T) = 1$.

- $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. mixed strategy profile $(\alpha_i^*, \alpha_{-i}^*)$

A **mixed strategy Nash equilibrium** is a mixed strategy profile α^* in a strategic game with vNM preferences such that

$\forall i \quad : \quad U_i(\alpha^*) \geq U_i(\alpha_i, \alpha_{-i}^*)$ with $\alpha_i \neq \alpha_i^*$

with $U_i(\alpha)$ the expected payoff of player i.

Example :

		pl 2: L (q)	R ($1-q$)
pl 1	T (p)	pq	$p(1-q)$
	B ($1-p$)	$(1-p)q$	$(1-p)(1-q)$

$$\alpha_1(T) = p$$
$$\alpha_1(B) = 1-p$$
$$\alpha_2(L) = q$$
$$\alpha_2(R) = 1-q$$

Expected payoff for player 1 :

$$pq\,u_1(T,L) + p(1-q)\,u_1(T,R) + (1-p)q\,u_1(B,L) + (1-p)(1-q)\,u_1(B,R)$$

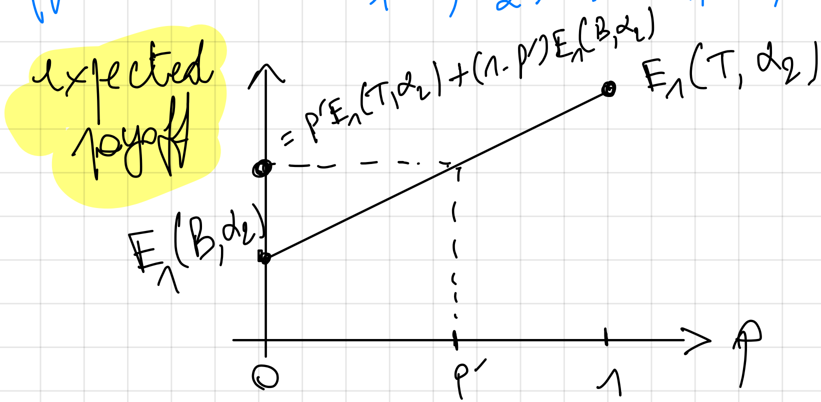
$$= p\left[q\,u_1(T,L) + (1-q)\,u_1(T,R)\right] + (1-p)\left[q\,u_1(B,L) + (1-q)\,u_1(B,R)\right]$$

$$= \boxed{p\,\underbrace{E_1(T,\alpha_2)}_{\text{expected payoff of playing } T} + (1-p)\,\underbrace{E_1(B,\alpha_2)}_{\text{expected payoff of playing } B}}$$

when pl 2 plays α_2 when pl 2 plays α_2.

$\Rightarrow$ the expected payoff of player 1 is a linear function of P.

Suppose that $E_1(T, \alpha_2) > E_1(B, \alpha_2)$



Then player 1 plays T with certainty ($P = 1$).

$\rightarrow$ no N.E in mixed strategy. , because pl 1 uses a pure strategy.

Conclusion : in a mixed strategy Nash equilibrium, We need $E_1(B, \alpha_2) = E_1(T, \alpha_2)$.