Université Paris 1 Panthéon Sorbonne MMMEF
Stochastic Approximation Algorithm
Noufel Frikha

Practical session: Discretisation of Stochastic Differential Equations

The aim of this practical session is to study and numerically illustrate the theoretical results presented in the course concerning the discretisation of stochastic differential equations (SDEs for short), using the Euler–Maruyama scheme and the Milstein scheme.

We work in the following framework. Let us consider a uniform partition of the interval $[0, T]$ into N subintervals of equal length, and define

$$t_k = k\frac{T}{N}, \quad k \in \{0, \dots, N\}.$$

The mesh size is denoted by $\Delta = T/N$. The Euler scheme consists in discretising the stochastic differential equation

$$X_t = x + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s, \quad t \in [0, T],$$

according to the time step Δ, by defining the process X^Δ as

$$X^\Delta_{t_{k+1}} = X^\Delta_{t_k} + b(t_k, X^\Delta_{t_k})\, \Delta + \sigma(t_k, X^\Delta_{t_k})(W_{t_{k+1}} - W_{t_k}), \quad X^\Delta_0 = x_0 \in \mathbb{R}^d.$$

This scheme is very easy to implement, since it only requires simulating the increments $(W_{t_{k+1}} - W_{t_k})_{0 \le k \le N-1}$, which are i.i.d. Gaussian random variables with distribution $\mathcal{N}(0, \Delta I_d)$.

In this practical session, we restrict ourselves to the case where the coefficients of the SDE are time-independent, that is $b(t, x) = b(x)$ and $\sigma(t, x) = \sigma(x)$. Under the assumption

$$\exists C_{b,\sigma} > 0,\ \forall (x, y) \in (\mathbb{R}^d)^2, \quad |b(x) - b(y)| + |\sigma(x) - \sigma(y)| \le C_{b,\sigma}|x - y|, \tag{1}$$

it is known that the *strong error* is of order $\sqrt{\Delta}$. More precisely, for all $p \ge 1$, there exists a constant $K(b, \sigma, T, p) > 0$ such that

$$\mathbb{E}\left[\left(\sup_{0 \le k \le N} |X_{t_k} - X^\Delta_{t_k}|\right)^p\right]^{1/p} \le K(b, \sigma, T, p)\sqrt{\Delta}.$$

To *numerically visualise* the discretisation error, it is convenient to consider the Black–Scholes model, whose dynamics are given by

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t, \qquad S_0 = x \in \mathbb{R}^*_+,$$

whose explicit solution is

$$S_t = x \exp\left((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\right),$$

and whose distribution is known exactly for all $t \in [0, T]$.

1 Study of the strong error

We fix $T = 1$, $\sigma = 40\%$, and $\mu = 10\%$.

1.1 Euler scheme

1. What is the theoretical behaviour of the quantities

$$\mathbb{E}[|S_T - S_T^\Delta|] \quad \text{and} \quad \mathbb{E}[|S_T - S_T^\Delta|^2]?$$

2. Using simulations, illustrate the behaviour of these quantities as functions of N. To this end, one may study Monte Carlo approximations of

$$N^p \mathbb{E}[|S_T - S_T^\Delta|] \quad \text{or} \quad N^q \mathbb{E}[|S_T - S_T^\Delta|^2],$$

as functions of N, where p and q are positive real numbers chosen appropriately.

1.2 Milstein scheme

The Milstein scheme is of higher order than the Euler scheme and yields a strong error (in $L^p(\mathbb{P})$) of order $\Delta = T/N$.

In dimension $d = 1$, we recall that in $L^2(\mathbb{P})$,

$$\mathbb{E}\left[\left(\int_{t_k}^{t_{k+1}} b(X_t)\,dt\right)^2\right] \leq \Delta \mathbb{E}\left[\int_{t_k}^{t_{k+1}} b(X_t)^2\,dt\right] \leq C_{b,\sigma,T}\Delta^2,$$

$$\mathbb{E}\left[\left(\int_{t_k}^{t_{k+1}} \sigma(X_t)\,dW_t\right)^2\right] = \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \sigma^2(X_t)\,dt\right] \leq C_{b,\sigma,T}\Delta.$$

Thus, for small Δ, the stochastic integral dominates the strong error. To improve accuracy, we discretise this term more precisely. Keeping only the leading-order terms in Δ, we write

$$\sigma(X_t) \approx \sigma(X_{t_k}) + \sigma'(X_{t_k})(X_t - X_{t_k}) \approx \sigma(X_{t_k}) + \sigma'(X_{t_k})\sigma(X_{t_k})(W_t - W_{t_k}).$$

This leads to the approximation

$$\int_{t_k}^{t_{k+1}} \sigma(X_t)\,dW_t \approx \sigma(X_{t_k})(W_{t_{k+1}} - W_{t_k}) + \frac{1}{2}\sigma'(X_{t_k})\sigma(X_{t_k})\left((W_{t_{k+1}} - W_{t_k})^2 - \Delta\right).$$

Assuming sufficient regularity of b and σ, the Milstein scheme $\bar{X}$ is defined by

$$\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + b(\bar{X}_{t_k})\Delta + \sigma(\bar{X}_{t_k})(W_{t_{k+1}} - W_{t_k}) + \frac{1}{2}\sigma'(\bar{X}_{t_k})\sigma(\bar{X}_{t_k})\left((W_{t_{k+1}} - W_{t_k})^2 - \Delta\right),$$

with $\bar{X}_0 = x_0$.

If b and σ are $\mathcal{C}^2$ functions with bounded derivatives, then, for all $p \geq 1$

$$\exists K(b,\sigma,T,p) > 0, \quad \mathbb{E}\left[\left(\sup_{0 \leq k \leq N} |X_{t_k} - \bar{X}_{t_k}|\right)^p\right]^{1/p} \leq K(b,\sigma,T,p)\frac{T}{N}.$$

1. Write explicitly the Milstein scheme $\bar{S}$ associated with the Black–Scholes model.
2. Show that for all $\alpha \in [0, 1)$,
$$N^{\alpha} \sup_{0 \leq k \leq N} |X_{t_k} - \bar{X}_{t_k}| \longrightarrow 0 \quad \text{a.s. as } N \to \infty,$$
and illustrate this result numerically.
3. What is the theoretical behaviour (as a function of N) of
$$\mathbb{E}[|S_T - \bar{S}_T|] \quad \text{and} \quad \mathbb{E}[|S_T - \bar{S}_T|^2]?$$
4. Illustrate these behaviours numerically as functions of N.

2 Study of the weak error

In this section, we study the *weak discretisation error*. For a Lipschitz function f, it is defined as
$$\mathcal{E}_f = \mathbb{E}[f(X_T)] - \mathbb{E}[f(X_T^{\Delta})].$$
This error arises when pricing a derivative with payoff $f(X_T)$, where the exact random variable X_T is replaced by its Euler approximation.

Using the strong error, one has the crude bound
$$\left|\mathbb{E}[f(X_T) - f(X_T^{\Delta})]\right| \leq C\mathbb{E}[|X_T - X_T^{\Delta}|] = \mathcal{O}\left(\sqrt{\frac{T}{N}}\right).$$

However, this estimate is very rough. It can be shown (see the course) that if b and σ are $\mathcal{C}^{\infty}$ with bounded derivatives, and if f is bounded with derivatives of polynomial growth, then
$$\mathbb{E}[f(X_T)] - \mathbb{E}[f(X_T^{\Delta})] = \frac{K_1}{N} + \frac{K_2}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right).$$

We consider a European call option with payoff $f(x) = e^{-rT}(x - K)_+$ under the risk-neutral Black–Scholes model
$$dS_t = rS_t\, dt + \sigma S_t\, dW_t, \qquad S_0 = x \in \mathbb{R}_+^*.$$

In this case, the Black-Scholes formula gives
$$\mathbb{E}[f(S_T)] = x\Phi(d_+(T, x, Ke^{-rT})) - Ke^{-rT}\Phi(d_-(T, x, Ke^{-rT})),$$
where
$$d_{\pm}(s, x, y) := \frac{1}{\sigma\sqrt{s}} \log\left(\frac{x}{y}\right) \pm \frac{1}{2}\sigma^2 s,$$
and Φ denotes the standard normal cumulative distribution function.

We take $S_0 = K = 100$, $\sigma = 40\%$, and $r = 5\%$. To compute Φ, we use the `erf` function from the `cmath` library, defined by
$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

```
#include <cmath>

double phi(double x1, double x2)
{
    return 0.5*(erf(x2/sqrt(2)) - erf(x1/sqrt(2)));
}
```

1. What is the theoretical behaviour of

$$\left|\mathbb{E}[f(S_T)] - \mathbb{E}[f(S_T^{\Delta})]\right|?$$

2. Illustrate this behaviour numerically by studying

$$N^p \left|\mathbb{E}[f(S_T) - f(S_T^{\Delta})]\right|$$

   as a function of N, for a suitable integer p.

3. Repeat the previous question for a put option with payoff $f(x) = e^{-rT}(K - x)_+$ and digital option.