March 12th 2026

Player 2

Player 1		L $\quad q$	R $\quad 1-q$
	T $\quad p$	$u_1(T,L), u_2(T,L)$	$u_1(T,R); u_2(T,R)$
	B $\quad 1-p$	$u_1(B,L), u_2(B,L)$	$u_1(B,R), u_2(B,R)$

$$\boxed{\alpha = \text{profile}} = (\alpha_1, \alpha_2)$$

$$\alpha_1 = (\text{Prob}(T); \text{Prob}(B)) = (\alpha_1(T); \alpha_1(B))$$
$$= (p, 1-p)$$

$$\alpha_2 = (\text{Prob}(L); \text{Prob}(R)) = (\alpha_2(L); \alpha_2(B))$$
$$= (q, 1-q)$$

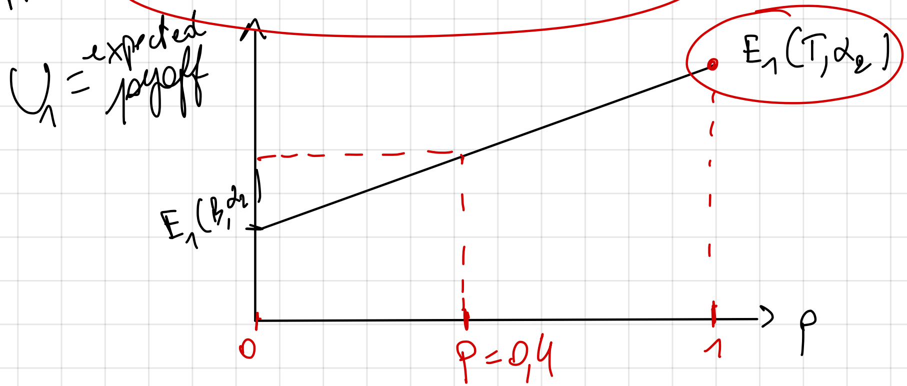
$U_1(\alpha)$ = expected payoff for player 1 confronted to the profile α of mixed strategies.

$$= p\,[q\,u_1(T,L) + (1-q)\,u_1(T,R)] + (1-p)\,[q\,u_1(B,L) + (1-q)\,u_1(B,R)]$$

$$= p \times \text{expected payoff } (T, \alpha_2) + (1-p) \times \text{expected payoff } (B, \alpha_2)$$

$$= p \times E_1(T, \alpha_2) + (1-p) \times E_1(B, \alpha_2)$$

Suppose $E_1(B, \alpha_2) < E_1(T, \alpha_2)$



In this case, player chooses $p = 1$, she plays T for sure.

$\rightarrow$ Pure strategy.

- When $E_1(B, \alpha_2) \neq E_1(T, \alpha_2) \rightarrow$ No NE in mixed strategies.

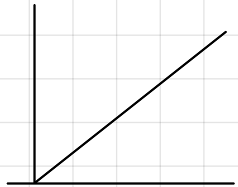
- In order to have a NE in mixed strategies we need

$$E_1(B, \alpha_2) = E_1(T, \alpha_2).$$

Affine

$ax + b$

Linear



ax

Several variables $y = f(x_1, x_2, \dots, x_n)$

f is linear if $a_1 x_1 + a_2 x_2 + \dots + a_n x_n$

$y = f(x_1, x_2)$

$y = a_1 x_1 + a_2 x_2$ $\rightarrow$ linear combination

$U_1 = p E_1(B, \alpha_2) + (1-p) E_1(T, \alpha_2)$

Example : Matching pennies.

		H (q)	T ($1-q$)
1	H (p)	$1,-1$	$-1,1$
	T ($1-p$)	$-1,1$	$1,-1$

$\alpha_1 = (p, 1-p)$

$\alpha_2 (q, 1-q)$

pl 1 H : $E_1(H, \alpha_2) = q(1) + (1-q)(-1) = 2q - 1$

$q\,u_1(H,H)$ $\quad u_1(H,T)$

pl 1 T : $E_1(T, \alpha_2) = q(-1) + (1-q)(1) = 1 - 2q$

$$U_1(\alpha) = U_1(\alpha_1, \alpha_2) = p\,E_1(H,\alpha_2) + (1-p)\,E_1(T,\alpha_2)$$

suppose $2q - 1 > 1 - 2q$

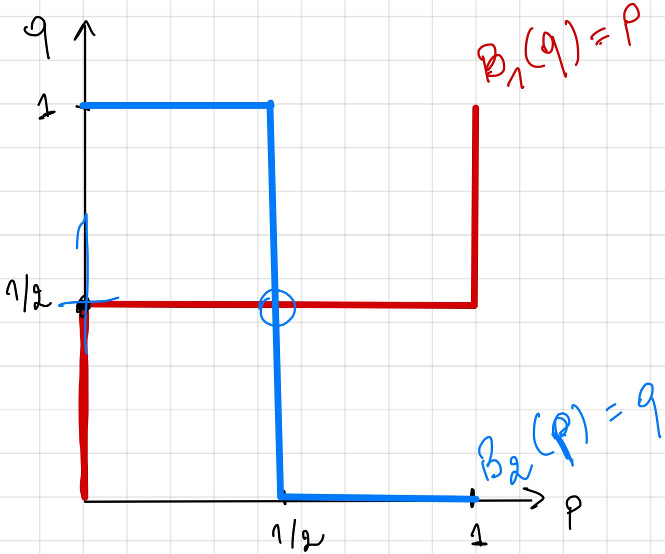
$\iff 4q > 2 \iff q > \frac{1}{2}$

if $q > 1/2$, $E_1(T,\alpha_2) > E_1(H,\alpha_2)$ and player 1 chooses $p = 1$, she plays T for sure.

$$\begin{cases} \text{if } q < 1/2 \rightarrow \text{she chooses } p=0 \text{, she plays H for sure,} \\ \text{if } q = 1/2 \rightarrow \text{she may choose any } p \in [0;1] \end{cases}$$

$$B_1(\alpha_2) = \begin{cases} p = 0 & \text{if } q < 1/2 \\ p \in [0;1] & \text{if } q = 1/2 \\ p = 1 & \text{if } q > 1/2 \end{cases}$$

$$B_2(\alpha_1 = p) = \begin{cases} q = 1 \text{ if } p < 1/2 \\ q \in [0;1] \text{ if } p = \frac{1}{2} \\ q = 0 \text{ if } p > \frac{1}{2} \end{cases}$$



they cross at only one point N.E in mixed strategies

$\alpha^* = (\alpha_1^*, \alpha_2^*)$

with $\alpha_1^* = (\frac{1}{2}, \frac{1}{2})$

and $\alpha_2^* = (\frac{1}{2}, \frac{1}{2})$.

$U_1 = \cancel{p} E_1(X) + \cancel{(1-p)} E_1(Y) \Rightarrow E_1(X) = E_1(Y)$

$a E_1(X) + b E_1(Y) + c E_1(Z)$

Example: Battle of the sexes.

		② B q	S $1-q$
①	B p	2, 1	0, 0
	S $1-p$	0, 0	1, 2

Player 1: $E_1(B, \alpha_2) = q \times 2 + (1-q)0 = 2q$

$E_1(S, \alpha_2) = q \times 0 + (1-q)1 = 1-q$

Construct the best response function

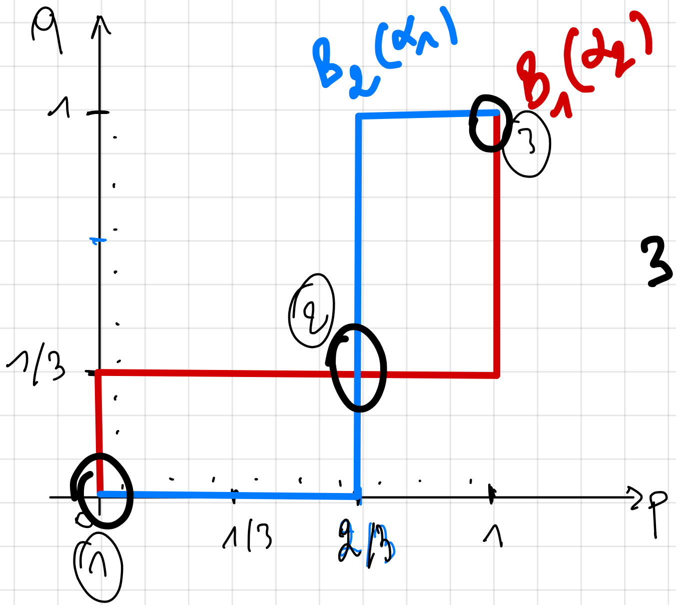
Suppose $E_1(B, \alpha_2) > E_1(S, \alpha_2)$

$2q > 1-q \iff 3q > 1$

$\iff q > 1/3$

When $q > 1/3$, player 1 chooses $p = 1$ (plays B for sure).
when $q < 1/3$, " " " $p = 0$ (plays S for sure)
when $q = 1/3$, " " " $p \in [0;1]$.

$$U_1(\alpha) = p\,E_1(B,\alpha_2) + (1-p)E_1(S,\alpha_2)$$

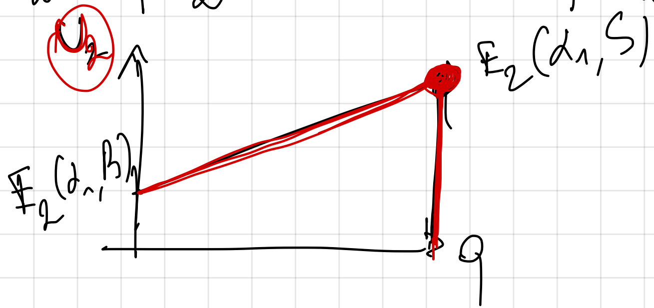


3 Nash equilibria.

Player 2 : $E_2(\alpha_1, B) = p \times 1 + (1-p) \times 0 = p$

$$E_2(\alpha_1, S) = p \times 0 + (1-p) \times 2 = 2-2p$$

$$\max_{q} U_2 = q\, E_2(\alpha_1, B) + (1-q)\, E_2(\alpha_1, S)$$



$$E_2(\alpha_1, S) > E_2(\alpha_1, B)$$

Suppose $2-2p > p \iff 2 > 3p \iff p < \frac{2}{3} \to q = \cancel{1}\ 0$

when $p < \frac{2}{3}$ player 2 chooses $q = \cancel{1}\ 0$

$B_2(\alpha_1)$ $p > \frac{2}{3}$ player 2 chooses $q = \cancel{0}\ 1$

| $p = 2/3$ player 2 chooses $q \in [0;1]$.

Best response functions intersect at 3 ≠ points:

(1) $\alpha_1^* = (0;1)$ and $\alpha_2^* = (0;1)$.

(2) $\alpha_1^* = \left(\frac{2}{3};\frac{1}{3}\right)$ and $\alpha_2^* = \left(\frac{1}{3};\frac{2}{3}\right)$

(3) $\alpha_1^* = (1;0)$ and $\alpha_2^* (1;0)$.

2 N.E in pure strategy.

New NE! where player 1 chooses B with proba $\frac{2}{3}$ in mixed strategies. and player 2 chooses B with proba $\frac{1}{3}$.

4.3.4. Characterization

the proba that player i assigns to action $a_i \in A_i$ in α_i her mixed strategy.

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i)\, E_i(a_i, \alpha_{-i})$$

$A_1 = (B, S)$

$a_1 = B$

if α^* is mixed strategy N.E.

$$\Rightarrow U_i(\alpha^*) = E_i^*$$

1) if $\alpha_i(a_i) > 0 \rightarrow E_i(a_i, \alpha^*_{-i}) = U_i(\alpha^*)$

2) if $\alpha_i(a_i') = 0 \rightarrow E_i(a_i', \alpha^*_{-i}) \leq U_i(\alpha^*)$.

	L_{0}	$C_{1/3}$	$R_{2/3}$
$T_{3/4}$	$\cdot, 2$	$3, 3$	$1, 1$
M_{0}	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$
$B_{1/4}$	$\cdot, 4$	$5, 1$	$0, 7$

$$E_1(T, \alpha_2) = E_1(B, \alpha_2) = \frac{5}{3}$$

$$E_1(M, \alpha_2) < 5/3$$

$$E_2(\alpha_1, C) = E_2(\alpha_1, R) = \frac{5}{2}$$

$$E_2(\alpha_1, L) < \frac{5}{2}$$