

Université Paris 1 Panthéon Sorbonne
MAEF, MMEF, QEM, IMAEF, 2024-2025

Final Exam: Portfolio Theory

14th May 2025

- Documents, calculators and cell phone are prohibited.
- The duration of the exam is **2h00**.
- It is forbidden to leave without returning the copy of your exam.
- The exam is composed of **3 exercices** which can be treated independently. In an exercise, you can use the results from the previous questions. We freely use the notations introduced in the course.

Exercise 1 (Questions on the lectures, 5pts)

1. (1pt) For a given loss L and confidence level $\alpha \in (0, 1)$, recall the definitions of the Value-at-Risk at level α of L denoted by $\text{VaR}_\alpha(L)$ and the Expected Shortfall at level α of L denoted by $\text{ES}_\alpha(L)$.
2. (1pt) Prove that for any $c \geq 0$ and $d \in \mathbb{R}$, one has

$$\text{VaR}_\alpha(cL + d) = c\text{VaR}_\alpha(L) + d.$$

3. (1pt) We now assume that $L \stackrel{d}{=} \mathcal{N}(\mu, \sigma^2)$ for some parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$. Prove that for any $\alpha \in (0, 1)$

$$\text{VaR}_\alpha(L) = \mu + \sigma\Phi^{-1}(\alpha).$$

where Φ is cumulative distribution function of $\mathcal{N}(0, 1)$.

4. (1pt) What are the axioms that a coherent risk measure must satisfy?
5. (1pt) Let $X = (X_1, \dots, X_N)$ be the return of N assets. Prove that the joint cdf $F(x) = \mathbb{P}(X_1 \leq x_1, \dots, X_N \leq x_N)$ with marginals $F_i(x_i) = \mathbb{P}(X_i \leq x_i)$ satisfies the following two sided bounds

$$C_{\min}(F_1(x_1), \dots, F_N(x_N)) \leq F(x) \leq C_{\max}(F_1(x_1), \dots, F_N(x_N))$$

where

$$C_{\min} = \max\left(\sum_{i=1}^N u_i + 1 - d, 0\right), \quad \text{and} \quad C_{\max}(u_1, \dots, u_N) = \min(u_1, \dots, u_N).$$

Exercise 2 (Markowitz portfolio theory, 10pts)

In this exercise, we assume that investors have mean-variance preferences. We follow the notations introduced in the course. We consider N risky assets. We denote by $R = (R_1, \dots, R_N)$ where R_i stands for the return of the asset i for $i = 1, \dots, N$. We assume that the expected returns $\mathbb{E}[R_i]$ are not all identical. The covariance matrix of R is denoted by Σ and is assumed to be invertible. The rate of return of the risk free asset is denoted by r . We also recall the following notations

$$a = \mathbf{1}^T \Sigma^{-1} \mathbb{E}[R], \quad b = \mathbb{E}[R]^T \Sigma^{-1} \mathbb{E}[R], \quad c = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$$

and finally $d = bc - a^2$ and $f^2 = b - 2ar + cr^2$. We also recall that $\mathbf{1} = (1, \dots, 1)$.

We consider the Lagrangian $\mathcal{L} = X^T \mathbb{E}[R] + (1 - \mathbf{1}^T X)r + \lambda(X^T \Sigma X - \sigma^2)$ where $\sigma^2 > 0$ is a given parameter.

1. (1pt) Prove that $d \geq 0$.
2. (1pt) Write explicitly the optimization problem to which the above Lagrangian \mathcal{L} is associated. What is its financial interpretation?
3. (1pt) Write explicitly the two first order conditions associated to the Lagrangian.
4. (1pt) Express the optimal weights X satisfying the two first order conditions as a function of the parameter λ .

5. (1pt) Prove that one has

$$2\lambda = \pm \frac{\sqrt{L}}{\sigma},$$

where L is a quantity you will determine. Justify that L is positive.

6. (1pt) Deduce the optimal weights X as a function of σ satisfying the optimality conditions of question 2.
7. (1pt) Compute the expected return of the optimal portfolio composed of risky assets only. You will distinguish the two cases $\lambda = \frac{1}{2} \frac{\sqrt{L}}{\sigma}$ and $\lambda = -\frac{1}{2} \frac{\sqrt{L}}{\sigma}$.
8. (1pt) Determine the fraction of wealth X_0 invested in the risk free asset. You will again distinguish the two cases $\lambda = \frac{1}{2} \frac{\sqrt{L}}{\sigma}$ and $\lambda = -\frac{1}{2} \frac{\sqrt{L}}{\sigma}$.
9. (1pt) Deduce the return μ as a function of σ of the optimal portfolio composed of N risky assets and the risk free asset. Again distinguish the two cases as previously done.
10. (1pt) When X is an efficient portfolio, give the expression of σ as a function of $\mu - r$ and give the expression of the weights $X(\mu)$ as a function of μ .

Exercise 3 (Risk measures, 5pts)

Let $X \in L^1(\mathbb{P})$. For x , we let $x_+ = \max(x, 0)$. For $\tau \in [0, 1]$, we define the map $\rho_\tau : L^1(\mathbb{P}) \rightarrow \mathbb{R}$ by

$$\rho_\tau(X) = \mathbb{E}[X] + (1 - \tau)\mathbb{E}[(X - \mathbb{E}[X])_+].$$

1. (1 pt) Prove that for any $c \geq 0$ and any $d \in \mathbb{R}$, it holds

$$\rho_\tau(cX + d) = c\rho_\tau(X) + d.$$

What is the name of the above property in the language of risk measures? What is its financial interpretation?

2. (1 pt) Prove that ρ_τ is sub-additive.
3. (1 pt) Show that if $X \leq 0$ then $\rho_\tau(X) \leq 0$.
4. (1 pt) Deduce from the previous question that ρ_τ is monotone.
5. (1 pt) Is ρ_τ a coherent risk measure for any $\tau \in [0, 1]$?