Consumer Theory

Preferences, demand, revealed preference, and choice under risk

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Course logistics

Assessment

- Midterm examination
- Final examination
- Attendance

Course materials

- Problem sets and recommended readings follow each lecture.
- Main reference: Mas-Colell, Whinston, and Green, *Microeconomic Theory*, Oxford University Press.
- Timetable and online resources are available on the course platform.

Course framework

Economic environment

A mathematical model describes commodities, technologies, institutions, and feasible actions.

Economic actors

The course focuses on consumers and producers.

Choices and interactions

- Consumption, production, and exchange
- Rational choice as an optimization problem
- Market interaction through general equilibrium

Lecture 1: commodities and preferences

Objective

Introduce the building blocks used to represent the choices of consumers and producers.

Reading: MWG 1A, 1B, 2A–2C, and 3B.

Commodities

- The economy contains a finite number of goods and services, indexed by $\ell = 1, \ldots, L$.
- A commodity is defined by its physical nature and, when relevant, its date, location, and state of availability.
- The first part of the course leaves time and uncertainty mostly implicit. Both become central in financial economics.

Key idea

Two physically identical goods delivered at different dates or in different states of the world count as different commodities.

Consumption bundles and feasible choices

Commodity vector

A bundle is a vector

$$x = (x_1, \ldots, x_L) \in \mathbb{R}^L,$$

where x_ℓ is the quantity of commodity ℓ.

Quantities may be fractional, so commodities are modeled as divisible.

Consumption set

$X \subseteq \mathbb{R}^L$ contains the bundles the consumer can actually choose, given physical, legal, and cultural constraints. (e.g. needs to survive so cannot consume 0)

We usually assume X is convex. For simplicity, the course often uses $X = \mathbb{R}^L_+$. But many of the proofs generalize to any convex consumption set.

$$x, y \in X,\ \lambda \in [0,1] \quad \Longrightarrow \quad \lambda x + (1-\lambda) y \in X.$$

Preference relations

Definition

A consumer's preferences are represented by a binary relation $\succeq$ on the consumption set X.

For any $x, y \in X$,

$$x \succeq y$$

means that the consumer considers x at least as good as y.

The relation allows either strict preference or indifference between bundles.

Examples of preference relations

Linear Given weights $a_\ell \geq 0$,

$$x \succeq_a y \iff \sum_{\ell=1}^{L} a_\ell x_\ell \geq \sum_{\ell=1}^{L} a_\ell y_\ell .$$

Leontief The least available component determines the ranking:

$$x \succeq_L y \iff \min_\ell x_\ell \geq \min_\ell y_\ell .$$

Lexicographic Compare the most important coordinate on which x and y differ. The bundle with the larger value at that coordinate ranks higher.

Contour and indifference sets

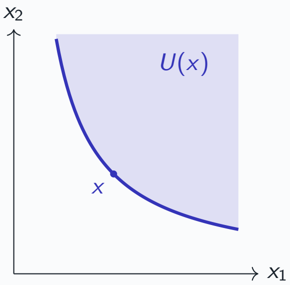
Fix $x \in X$. Preferences divide the consumption set into three regions.

$$\begin{aligned} I(x) &= \{y \in X : y \sim x\} && \text{indifference set,} \\ U(x) &= \{y \in X : y \succeq x\} && \text{upper contour set,} \\ L(x) &= \{y \in X : x \succeq y\} && \text{lower contour set.} \end{aligned}$$

Interpretation

$U(x)$ contains bundles at least as good as x; $L(x)$ contains bundles no better than x.

Upper and lower contour sets



upper contour set

lower contour set

The indifference curve $I(x)$ forms the common boundary for monotone preferences.

Strict preference and indifference

Starting from $\succeq$, define two auxiliary relations:

Strict preference

$$x \succ y \iff x \succeq y \text{ and } \neg(y \succeq x).$$

Indifference

$$x \sim y \iff x \succeq y \text{ and } y \succeq x.$$

Properties of binary relations

A binary relation R on a set E is:

- **Complete** for every $x, y \in E$, either xRy or yRx.
- **Transitive** xRy and yRz imply xRz.
- **Reflexive** xRx for every $x \in E$.
- **Irreflexive** $\neg(xRx)$ for every $x \in E$.
- **Symmetric** xRy implies yRx.

Equivalence relation

A relation that is reflexive, transitive, and symmetric.

Rational preferences

Definition

A preference relation is **rational** when it is complete and transitive.

Implications

If $\succeq$ is rational, then:

1. $\succeq$ is reflexive and therefore a complete preorder;
2. $\sim$ is an equivalence relation;
3. $\succ$ is irreflexive and transitive;
4. $x \succ y$ and $y \succeq z$ imply $x \succ z$.

Why transitivity can fail descriptively

Transitivity is analytically convenient, but observed choices can depart from it.

Perception

Small differences may be individually imperceptible but become visible when accumulated.

Social choice

Majority voting can generate cycles even when each voter is rational.

Framing

Equivalent outcomes may elicit different choices when described as gains or losses.

Framing effects

Another potential problem arises when the manner in which alternatives are presented matters for choice. This is known as the *framing* problem. Consider the following example, paraphrased from Kahneman and Tversky (1984):

> Imagine that you are about to purchase a stereo for 125 dollars and a calculator for 15 dollars. The salesman tells you that the calculator is on sale for 5 dollars less at the other branch of the store, located 20 minutes away. The stereo is the same price there. Would you make the trip to the other store?

It turns out that the fraction of respondents saying that they would travel to the other store for the 5 dollar discount is much higher than the fraction who say they would travel when the question is changed so that the 5 dollar saving is on the stereo. This is so even though the ultimate saving obtained by incurring the inconvenience of travel is the same in both cases.[3] Indeed, we would expect indifference to be the response to the following question:

> Because of a stockout you must travel to the other store to get the two items, but you will receive 5 dollars off on either item as compensation. Do you care on which item this 5 dollar rebate is given?

If so, however, the individual violates transitivity. To see this, denote

x = Travel to the other store and get a 5 dollar discount on the calculator.
y = Travel to the other store and get a 5 dollar discount on the stereo.
z = Buy both items at the first store.

The first two choices say that $x \succ z$ and $z \succ y$, but the last choice reveals $x \sim y$. Many problems of framing arise when individuals are faced with choices between alternatives that have uncertain outcomes (the subject of Chapter 6). Kahneman and Tversky (1984) provide a number of other interesting examples.

Condorcet cycles and social choice

Individual rankings

$$\begin{aligned} 1 &: \quad A \succ B \succ C \\ 2 &: \quad B \succ C \succ A \\ 3 &: \quad C \succ A \succ B \end{aligned}$$

Pairwise majority rule yields

$$A \succ B, \qquad B \succ C, \qquad C \succ A.$$

The social ranking cycles, even though each individual ranking is transitive. One

can always find a majority of agents against any given choice: social preferences can hardly be defined. Arrow impossibility theorem: generic impossibility to aggregate individual preference into a (transitive) social preference (unless aggregation rule is dictatorial).

Monotonicity and local nonsatiation

Let $X = \mathbb{R}^L_+$. A preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is said to be

- Monotone if for all $x, y \in X$ one has: $y > x \Rightarrow y \succ x$.
- Strongly monotone if for all $x, y \in X$ one has: $[y \geq x \wedge y \neq x] \Rightarrow y \succ x$.
- Locally nonsatiated if for all $x \in X$ and all $\epsilon > 0$ there exists $y \in X$ such that $\|y - x\| < \epsilon$ and $y \succ x$.

Logical relationship

Strong monotonicity $\Longrightarrow$ monotonicity $\Longrightarrow$ local nonsatiation.

Convex preferences

Convexity

Preferences are convex when upper contour sets are convex:

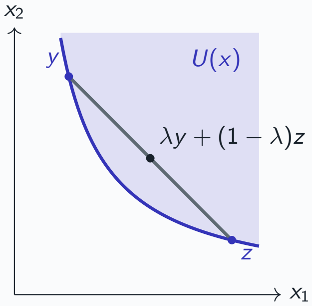
$$y \succeq x,\ z \succeq x \Longrightarrow \lambda y + (1-\lambda)z \succeq x \quad \forall \lambda \in [0,1].$$

Strict convexity

If $y \neq z$, strict convexity requires

$$\lambda y + (1-\lambda)z \succ x \quad \forall \lambda \in (0,1).$$

Convexity of an upper contour set



Geometric test

Pick any $y, z \in U(x)$. Every point on the line segment between them must also lie in $U(x)$.

$$y \succeq x,\ z \succeq x \quad \Rightarrow \quad \lambda y + (1-\lambda)z \succeq x.$$

Strict convexity places interior mixtures strictly above the boundary when $y \neq z$.

Economic meaning of convexity

Taste for diversification

A mixture of two bundles that are each at least as good as x remains at least as good as x.

Diminishing marginal rate of substitution

it takes increasingly larger amount of one commodity (bundle) to compensate for losses of another:

- Assume $\succsim$ strictly convex and $x, y \in X$ such that $x + \delta \sim x$ (think $\delta = a - b$).
- One has $x + \delta = \frac{1}{2}(x + 2\delta) + \frac{1}{2}x$.
- One must have $x \succ x + 2\delta$ as otherwise one would have $x + \delta \succ x$.

Continuity of preferences

Sequential definition

A preference relation $\succeq$ on $X = \mathbb{R}^L_+$ is continuous if, for any convergent sequences (x_n) and (y_n),

$$x_n \succeq y_n \quad \forall n \quad \Longrightarrow \quad \lim_{n\to\infty} x_n \succeq \lim_{n\to\infty} y_n.$$

Continuity excludes abrupt reversals in rankings under arbitrarily small changes in bundles.

Worked example: linear and Leontief preferences

Let $X = \mathbb{R}^2_+$ and compare two preference relations.

Perfect substitutes

For $a \gg 0$, define

$$x \succeq_S y \iff a \cdot x \geq a \cdot y.$$

Complete, transitive, continuous, strongly monotone, and convex, but not strictly convex.

Perfect complements

Let $m(x) = \min\{x_1, \beta x_2\}$, $\beta > 0$. Then

$$x \succeq_C y \iff m(x) \geq m(y).$$

Complete, transitive, continuous, monotone, and convex, but not strongly monotone.

Worked example: lexicographic preferences

On $X = \mathbb{R}^2_+$, define

$$x \succeq_L y \iff \big[x_1 > y_1\big] \quad \text{or} \quad \big[x_1 = y_1 \text{ and } x_2 \geq y_2\big].$$

- The relation is complete, transitive, and strongly monotone.
- It is convex: mixtures of bundles in an upper contour set stay in that set.
- It is not continuous. For $x = (1, 0)$, $y = (0, 1)$, and $x^n = (1 - 1/n, 1)$, we have $x \succ x^n$ for every n, although $x^n \to (1, 1) \succ x$.

Application exercise: rationed complements

A household chooses heating fuel x_1 and electricity x_2. Technical constraints make the services complementary. Define

$$x \succeq y \iff \min\{x_1, 2x_2\} \geq \min\{y_1, 2y_2\}.$$

1. Prove that $\succeq$ is rational and continuous.
2. Characterize $U(x)$ and $I(x)$ for an arbitrary $x \in \mathbb{R}^2_+$.
3. Determine whether $\succeq$ is monotone, strongly monotone, locally nonsatiated, convex, and strictly convex. Justify each answer.
4. A regulation imposes $x_1 \leq \bar{x}_1$. Explain how the restriction changes the set of undominated feasible bundles when electricity remains unrestricted.

Lecture 2: utility functions

Objective

Introduce utility representations and formulate the consumer's utility maximization problem.

Reading: MWG 1B and 3C.

Utility representation

Utility function

A utility function is a mapping $u : X \to \mathbb{R}$.

It induces a preference relation:

$$x \succeq_u y \iff u(x) \geq u(y).$$

Conversely, u **represents** a preference relation $\succeq$ when

$$x \succeq y \iff u(x) \geq u(y) \qquad \text{for every } x, y \in X.$$

Utility numbers encode rankings; their absolute magnitudes need not carry meaning.

Common utility functions

Linear

$$u(x) = \sum_{\ell=1}^{L} a_\ell x_\ell, \qquad a_\ell \geq 0.$$

Leontief

$$u(x) = \min\{a_1 x_1, \ldots, a_L x_L\}, \qquad a_\ell > 0.$$

Cobb–Douglas

$$u(x) = \prod_{\ell=1}^{L} x_\ell^{a_\ell}, \qquad a_\ell \geq 0, \quad \sum_{\ell=1}^{L} a_\ell = 1.$$

CES utility functions

Definition

A constant elasticity of substitution utility function takes the form

$$u_\rho(x) = \left(\sum_{\ell=1}^{L} a_\ell x_\ell^\rho \right)^{1/\rho}, \qquad a_\ell \geq 0,$$

with elasticity of substitution $\sigma = 1/(1-\rho)$.

- $\rho = 1$: linear utility.
- $\rho \to 0$: Cobb–Douglas limit when $\sum_\ell a_\ell = 1$.
- $\rho \to -\infty$: Leontief limit under the usual CES parameterization.

CES limit: Cobb–Douglas, step 1

For two goods, consider

$$\lim_{\rho\to 0}(a_1 x_1^{\rho} + a_2 x_2^{\rho})^{1/\rho}, \qquad a_1 + a_2 = 1.$$

Taking logarithms gives

$$L = \lim_{\rho\to 0} \frac{\log(a_1 x_1^{\rho} + a_2 x_2^{\rho})}{\rho}.$$

L'Hôpital's rule yields

$$L = \lim_{\rho\to 0} \frac{a_1 x_1^{\rho} \log x_1 + a_2 x_2^{\rho} \log x_2}{a_1 x_1^{\rho} + a_2 x_2^{\rho}}.$$

CES limit: Cobb–Douglas, step 2

Since $x_i^\rho \to 1$ as $\rho \to 0$,

$$L = a_1 \log x_1 + a_2 \log x_2 = \log(x_1^{a_1} x_2^{a_2}).$$

Exponentiating both sides gives

$$\boxed{\lim_{\rho \to 0} (a_1 x_1^\rho + a_2 x_2^\rho)^{1/\rho} = x_1^{a_1} x_2^{a_2}}$$

when $a_1 + a_2 = 1$.

CES limit: Leontief, step 1

Use the parameterization

$$v_r(x) = (a_1^r x_1^r + a_2^r x_2^r)^{1/r}, \qquad r < 0.$$

We seek the limit as $r \to -\infty$. Taking logarithms,

$$L = \lim_{r \to -\infty} \frac{\log(a_1^r x_1^r + a_2^r x_2^r)}{r}.$$

This is the negative-order power mean, which converges to the smaller effective quantity.

CES limit: Leontief, step 2

Suppose, without loss of generality, that $a_1x_1 < a_2x_2$. Factor out $(a_1x_1)^r$:

$$v_r(x) = a_1x_1\left[1 + \left(\frac{a_2x_2}{a_1x_1}\right)^r\right]^{1/r}.$$

Since $(a_2x_2)/(a_1x_1) > 1$ and $r \to -\infty$, the ratio raised to r converges to zero. Therefore,

$$\boxed{\lim_{r\to-\infty} v_r(x) = a_1x_1 = \min\{a_1x_1, a_2x_2\}.}$$

Ordinal and cardinal properties

Monotone transformations

If u represents $\succeq$ and $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $f \circ u$ represents the same preferences.

- An **ordinal** property is invariant under increasing transformations. Example: whether x is preferred to y.
- A **cardinal** property depends on the numerical scale. Example: whether one bundle has twice the utility of another.

For Cobb–Douglas utility,

$$x_1^{\alpha} x_2^{1-\alpha} \quad \text{and} \quad \alpha \log x_1 + (1-\alpha) \log x_2$$

represent the same preferences on $\mathbb{R}^2_{++}$.

When does a utility representation exist?

Necessary condition

Any preference relation represented by a utility function must be rational.

Rationality alone is not sufficient on arbitrary sets. Lexicographic preferences on $\mathbb{R}^2$ provide a standard counterexample.

Why lexicographic preferences fail

A real-valued utility representation would require a strictly ordered assignment of distinct rational intervals to uncountably many first-coordinate values, which is impossible.

Utility representation theorem

Continuity

If $x_n \succeq y_n$ for every n and both sequences converge, then their limits preserve the ranking.

Theorem

A rational and continuous preference relation on $\mathbb{R}^L_+$ can be represented by a continuous utility function.

Completeness and transitivity deliver coherent rankings; continuity allows those rankings to be encoded by real numbers.

Proof idea for monotone preferences

Assume $\succeq$ is monotone, continuous, complete, and transitive.

1. Let $e = (1, \ldots, 1) \in \mathbb{R}^L_+$.
2. For every $x \in \mathbb{R}^L_+$, continuity and monotonicity yield a scalar $\alpha(x)$ such that
$$x \sim \alpha(x)e.$$
3. The function $\alpha : \mathbb{R}^L_+ \to \mathbb{R}$ represents $\succeq$.
4. Continuity of preferences implies continuity of α.

The construction assigns each bundle the amount of the equal-components bundle that makes the consumer indifferent.

Monotonicity properties of utility

A utility function inherits the corresponding properties of the preference relation it represents. A utility function u is

Monotone $y > x \Rightarrow u(y) > u(x)$.

Strongly monotone $y \geq x$ and $y \neq x \Rightarrow u(y) > u(x)$.

Locally nonsatiated For every x and $\varepsilon > 0$, some y satisfies

$$\|y - x\| < \varepsilon \quad \text{and} \quad u(y) > u(x).$$

Continuity and quasiconcavity

- u is continuous if and only if the represented preference relation $\succeq_u$ is continuous.
- $\succeq_u$ is convex if and only if u is quasiconcave:

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}, \qquad \lambda \in [0,1].$$

- $\succeq_u$ is strictly convex if and only if u is strictly quasiconcave:

$$u(\lambda x + (1-\lambda)y) > \min\{u(x), u(y)\}, \qquad x \neq y,\ \lambda \in (0,1).$$

Application exercise: utility representations

On $X = \mathbb{R}^2_{++}$, consider

$$u(x) = x_1^{\alpha} x_2^{1-\alpha}, \qquad v(x) = \log u(x), \qquad z(x) = u(x)^2, \quad \alpha \in (0,1).$$

1. Prove that u, v, and z represent the same preference relation.
2. Determine which functions are concave and which are quasiconcave.
3. Show directly that the represented preference relation is strictly convex and strongly monotone.
4. Explain why concavity is not an ordinal property, while quasiconcavity is.

Lecture 3: consumer choice and demand

Objective

Study utility maximization under a budget constraint and derive the main properties of Walrasian demand.

Reading: MWG 2D, 3D, and 2F.

Prices and price-taking behavior

A complete market system posts one price for each commodity.

- The price vector is $p = (p_1, \ldots, p_L) \in \mathbb{R}^L_+ \setminus \{0\}$.
- Consumers treat prices as given and cannot influence them individually.
- p_ℓ measures the units of account required for one unit of commodity ℓ.

Value of a bundle

The expenditure on $x \in \mathbb{R}^L_+$ is the inner product $p \cdot x = \sum_{\ell=1}^{L} p_\ell x_\ell$.

The budget set

A consumer with wealth $w \geq 0$ who faces prices p can afford

$$B(p, w) = \{x \in X : p \cdot x \leq w\}.$$

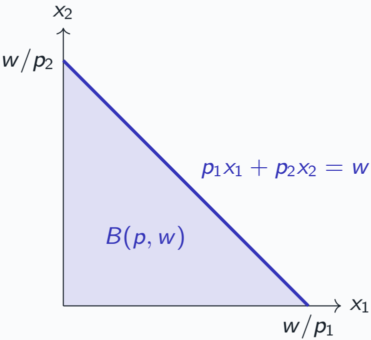
For $X = \mathbb{R}^L_+$ and $p \gg 0$, the budget frontier is

$$\{x \geq 0 : p \cdot x = w\}.$$

Two goods

The slope of the budget line is $-p_1/p_2$. Its intercepts are w/p_1 and w/p_2.

The budget set in two dimensions



Affordable bundles

The shaded triangle contains all $x \geq 0$ satisfying $p \cdot x \leq w$.

- Slope: $-p_1/p_2$
- Horizontal intercept: w/p_1
- Vertical intercept: w/p_2

Properties of the budget set

Let $X = \mathbb{R}^L_+$, $p \in \mathbb{R}^L_+ \setminus \{0\}$, and $w \geq 0$.

1. $B(p, w)$ is nonempty, closed, and convex.
2. $B(p, w)$ is bounded when $p \gg 0$.
3. For every $t > 0$,
$$B(tp, tw) = B(p, w).$$

Interpretation

Multiplying all prices and wealth by the same factor changes the unit of account, not the feasible choices.

The utility maximization problem

Given $u : X \to \mathbb{R}$, the consumer solves

$$\max_{x \geq 0} \ u(x) \qquad \text{subject to} \qquad p \cdot x \leq w.$$

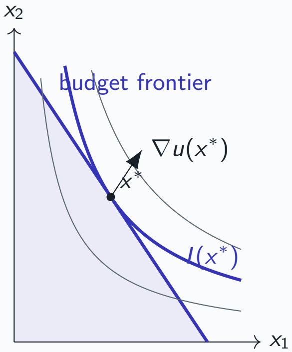
Walrasian demand $d(p, w)$ is the set of solutions.

Indirect utility $v(p, w) = u(x)$ for any $x \in d(p, w)$.

Demand can also be defined directly from preferences:

$$d(p, w) = \{x \in B(p, w) : x \succeq y \text{ for every } y \in B(p, w)\}.$$

Optimization over a budget set



Interior optimum

The highest attainable indifference curve is tangent to the budget frontier.

$$\nabla u(x^*) = \lambda p,$$
$$p \cdot x^* = w.$$

Hence

$$\text{MRS}_{12}(x^*) = p_1/p_2.$$

Properties of Walrasian demand

Suppose u is continuous and locally nonsatiated, with $p \gg 0$.

1. $d(p, w)$ is nonempty.
2. $d(tp, tw) = d(p, w)$ for every $t > 0$.
3. Every $x \in d(p, w)$ satisfies Walras' law: $p \cdot x = w$.
4. If preferences are convex, then $d(p, w)$ is convex.
5. If preferences are strictly convex, demand is single-valued. Under standard regularity conditions, it is continuous in (p, w).

Proof: existence and homogeneity

Existence

Since $p \gg 0$, every $x \in B(p, w)$ satisfies $0 \leq x_\ell \leq w/p_\ell$. Thus $B(p, w)$ is nonempty and compact. Continuity of u and the Weierstrass theorem imply that u attains a maximum on $B(p, w)$, so $d(p, w) \neq \varnothing$.

Homogeneity of degree zero

For every $t > 0$,

$$x \in B(tp, tw) \iff tp \cdot x \leq tw \iff p \cdot x \leq w.$$

Hence $B(tp, tw) = B(p, w)$. The two maximization problems have the same objective and feasible set, so $d(tp, tw) = d(p, w)$.

Proof: Walras' law

Take $x^* \in d(p, w)$ and suppose, toward a contradiction, that $p \cdot x^* < w$. Choose $\varepsilon > 0$ such that

$$p \cdot x^* + \|p\|\varepsilon < w.$$

Local nonsatiation provides $y \in X$ with $\|y - x^*\| < \varepsilon$ and $y \succ x^*$. By Cauchy–Schwarz,

$$p \cdot y \leq p \cdot x^* + \|p\|\|y - x^*\| < w.$$

Thus y is affordable and strictly preferred to x^*, contradicting optimality. Therefore every demanded bundle exhausts wealth:

$$\boxed{p \cdot x^* = w.}$$

Proof: convexity and uniqueness

Let $x, y \in d(p, w)$ and $z_\lambda = \lambda x + (1 - \lambda)y$ for $\lambda \in (0, 1)$. Convexity of the budget set gives $z_\lambda \in B(p, w)$. Since x and y both maximize utility, $u(x) = u(y) = u^*$.

Convex preferences

Quasiconcavity gives

$$u(z_\lambda) \geq \min\{u(x), u(y)\} = u^*.$$

No feasible bundle can yield more than u^*, hence z_λ is also demanded. Therefore $d(p, w)$ is convex.

Strictly convex preferences

If $x \neq y$, strict quasiconcavity gives $u(z_\lambda) > u^*$, a contradiction. Thus demand contains at most one bundle.

Proof: continuity of single-valued demand

Suppose preferences are continuous and strictly convex, so demand is the function $x(p, w)$. Consider $(p^n, w^n) \to (p, w)$ with $p \gg 0$ and set $x^n = x(p^n, w^n)$.

1. Prices remain bounded away from zero near p, so the sequence (x^n) lies in a common compact set.
2. Any subsequence has a convergent further subsequence $x^{n_k} \to \bar{x}$.
3. Feasibility passes to the limit. The maximum theorem, or its closed-graph argument, implies $\bar{x} \in d(p, w)$.
4. Strict convexity makes $d(p, w) = \{x(p, w)\}$, so every convergent subsequence has the same limit $x(p, w)$.

Therefore $x(p^n, w^n) \to x(p, w)$. This is continuity on $\mathbb{R}^L_{++} \times \mathbb{R}_{++}$.

Price normalization

Homogeneity of degree zero means that only relative prices matter.

Two common normalizations

- Choose a numeraire: set one price, such as p_L, equal to one.
- Normalize the price simplex: impose $\sum_{\ell=1}^{L} p_\ell = 1$.

Normalization removes one redundant degree of freedom. It does not impose an economic restriction.

KKT characterization of demand

Assume u is differentiable. At an optimum x^*, there is a multiplier $\lambda \geq 0$ such that

$$\frac{\partial u(x^*)}{\partial x_\ell} \leq \lambda p_\ell, \qquad x_\ell^* > 0 \implies \frac{\partial u(x^*)}{\partial x_\ell} = \lambda p_\ell.$$

Complementary slackness also requires

$$\lambda(w - p \cdot x^*) = 0.$$

Under local nonsatiation, the budget binds and $\lambda > 0$ when marginal utility is positive.

Marginal rates of substitution

At an interior optimum with $\nabla u(x^*) \gg 0$,

$$\nabla u(x^*) = \lambda p.$$

Therefore, for every pair k, ℓ,

$$\mathrm{MRS}_{k\ell}(x^*) = \frac{\partial u(x^*)/\partial x_k}{\partial u(x^*)/\partial x_\ell} = \frac{p_k}{p_\ell}.$$

The consumer's local willingness to substitute equals the market's exchange rate between the goods.

When are the first-order conditions sufficient?

- If u is concave, the feasible set is convex and the KKT conditions are sufficient for a global optimum.
- For twice differentiable u, concavity requires a negative semidefinite Hessian.
- If u is differentiable and quasiconcave, with $\nabla u(x) \neq 0$, a tangency condition plus feasibility can also identify a global optimum under suitable boundary conditions.
- Strict quasiconcavity makes any optimum unique.

Duality for comparative statics

Assume preferences are continuous, locally nonsatiated, and strictly convex. Let $x(p, w)$ denote Walrasian demand and define

$$e(p, \bar{u}) = \min_{z \geq 0}\{p \cdot z : u(z) \geq \bar{u}\}, \qquad h(p, \bar{u}) = \arg\min\{p \cdot z : u(z) \geq \bar{u}\}.$$

The primal and dual problems imply

$$h(p, v(p, w)) = x(p, w), \qquad e(p, v(p, w)) = w.$$

$h(p, u)$ is called the compensated (or Hicksian) demand

Shephard's lemma

Fix a utility target $\bar{u}$ and define

$$e(p,\bar{u}) = \min_{x \geq 0}\{p \cdot x : u(x) \geq \bar{u}\}, \qquad h(p,\bar{u}) \in \arg\min\{p \cdot x : u(x) \geq \bar{u}\}.$$

Lemma

If compensated demand is locally single-valued and continuous at $(p,\bar{u})$, then e is differentiable with respect to prices and

$$\boxed{\nabla_p e(p,\bar{u}) = h(p,\bar{u}).}$$

Thus $\partial e/\partial p_i$ equals the compensated demand for good i.

Proof of Shephard's lemma

Let $h = h(p, \bar{u})$ and $h' = h(p', \bar{u})$. Optimality at the two price vectors gives

$$e(p', \bar{u}) \leq p' \cdot h = e(p, \bar{u}) + (p' - p) \cdot h,$$

and

$$e(p, \bar{u}) \leq p \cdot h' = e(p', \bar{u}) - (p' - p) \cdot h'.$$

Set $p' = p + tq$ with $t > 0$. Rearranging yields

$$q \cdot h(p + tq, \bar{u}) \leq \frac{e(p + tq, \bar{u}) - e(p, \bar{u})}{t} \leq q \cdot h(p, \bar{u}).$$

Continuity of h and $t \downarrow 0$ imply $D_p e(p, \bar{u}) q = q \cdot h(p, \bar{u})$ for every direction q. Hence $\nabla_p e(p, \bar{u}) = h(p, \bar{u})$.

Proof of the Slutsky equation

Start from the identity

$$h_i(p,\bar{u}) = x_i\big(p, e(p,\bar{u})\big).$$

Differentiate with respect to p_j, holding $\bar{u}$ fixed:

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}\frac{\partial e}{\partial p_j}.$$

Shephard's lemma gives $\partial e/\partial p_j = h_j$. Evaluating at $\bar{u} = v(p,w)$ yields $h_j = x_j(p,w)$, hence

$$\boxed{\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j\frac{\partial x_i}{\partial w}.}$$

Compensated substitution effects

Define the Slutsky matrix

$$S(p,w) = D_p x(p,w) + x_w(p,w)x(p,w)^\top .$$

The Slutsky equation implies

$$S(p,w) = D_p h\big(p, v(p,w)\big) = D^2_{pp} e\big(p, v(p,w)\big).$$

The expenditure function is concave in prices because it is the pointwise infimum of functions linear in p. Therefore its Hessian is symmetric and negative semidefinite:

$$S = S^\top, \qquad a^\top S a \leq 0 \quad \forall a \in \mathbb{R}^L .$$

In particular, compensated own-price effects satisfy $S_{ii} \leq 0$.

Proof of homogeneity and adding-up restrictions

Homogeneity of Walrasian demand gives $x(tp, tw) = x(p, w)$. Differentiating in t at $t = 1$ yields

$$D_p x(p, w)p + w\, x_w(p, w) = 0.$$

Walras' law gives $p^\top x(p, w) = w$. Differentiating with respect to wealth and prices gives

$$p^\top x_w(p, w) = 1, \qquad p^\top D_p x(p, w) = -x(p, w)^\top.$$

These equations restrict the joint responses of all goods even when individual cross-price effects remain ambiguous.

Income and own-price comparative statics

Setting $j = i$ in the Slutsky equation gives

$$\frac{\partial x_i}{\partial p_i} = S_{ii} - x_i \frac{\partial x_i}{\partial w}, \qquad S_{ii} \leq 0.$$

- If good i is normal, $\partial x_i / \partial w \geq 0$, so its Walrasian own-price effect is nonpositive.
- If good i is inferior, the income effect works against the substitution effect.
- A Giffen good requires the positive income-effect term $-x_i \, \partial x_i / \partial w$ to dominate the negative compensated effect S_{ii}.
- Compensated cross-price effects are symmetric. Walrasian cross-price effects need not be.

Worked example: corner and kink solutions

Perfect substitutes

For $u = x_1 + ax_2$, compare $1/p_1$ with a/p_2. Demand concentrates expenditure on the good with greater utility per euro. Equality produces a set of optima.

Perfect complements

For $u = \min\{x_1, bx_2\}$, efficiency requires $x_1 = bx_2$. Thus

$$x_2^* = \frac{w}{bp_1 + p_2}, \qquad x_1^* = \frac{bw}{bp_1 + p_2}.$$

Application exercise: CES demand

On $X = \mathbb{R}^2_{++}$, let

$$u(x) = \left[a x_1^\rho + (1-a) x_2^\rho\right]^{1/\rho}, \quad a \in (0,1), \quad \rho < 1, \quad \rho \neq 0,$$

and write $\sigma = 1/(1-\rho)$.

1. Prove homogeneity of degree one and strict quasiconcavity.
2. Derive $x_1/x_2 = [(a/(1-a))(p_2/p_1)]^\sigma$.
3. Obtain Walrasian demand and expenditure shares.
4. Study the limits $\rho \to 0$ and $\rho \to -\infty$.

Lecture 3bis: revealed preference

Objective

Infer restrictions on preferences from observed choices and assess whether demand data are consistent with rational choice.

Reading: MWG 1C, 1D, and 2F through page 30.

Why revealed preference?

Preferences are not directly observable. Economists only observe feasible sets and choices.

- A chosen bundle reveals that it was at least as good as every affordable alternative.
- Repeated choices restrict the preference relations that could rationalize behavior.
- The approach tests consistency with optimization without assigning utility numbers in advance.

Choice structures and choice rules

Choice structure

A family $\mathcal{B}$ of feasible subsets of X, together with a rule

$$c : \mathcal{B} \to 2^X, \qquad \varnothing \neq c(B) \subseteq B.$$

For consumer data,

$$\mathcal{B} = \{B(p,w) : p \in \mathbb{R}^L_+ \setminus \{0\},\ w \geq 0\}, \qquad c(B(p,w)) = d(p,w).$$

Directly revealed preference

If the consumer chooses x^t at (p^t, w^t), then

$$p^t \cdot y \leq w^t \quad \Longrightarrow \quad x^t \text{ is directly revealed at least as good as } y.$$

For observed choices with budget exhaustion, $w^t = p^t \cdot x^t$, so the affordability test becomes

$$p^t \cdot x^s \leq p^t \cdot x^t.$$

Revealed preference records implications of choice; it does not require observing utility.

The weak axiom of revealed preference

A single-valued demand function satisfies WARP if

$$p' \cdot x(p, w) \leq w' \quad \text{and} \quad x(p, w) \neq x(p', w')$$

imply

$$p \cdot x(p', w') > w.$$

If the new choice was available under the old budget, the old choice cannot also be available under the new budget unless the two choices coincide.

Rationalizability

A preference relation $\succeq$ induces the choice rule

$$c_{\succeq}(B) = \{x \in B : x \succeq z \text{ for every } z \in B\}.$$

- If $\succeq$ is rational, $c_{\succeq}$ satisfies WARP.
- A rule c is rationalizable when $c(B) = c_{\succeq}(B)$ for every observed feasible set.
- With a sufficiently rich collection of feasible sets, WARP can recover a unique rational ranking.

The role of the observable domain

WARP alone need not guarantee rationalizability when the family $\mathcal{B}$ is sparse.

Three-choice cycle

Let $X = \{x, y, z\}$ and observe only the pairs:

$$c(\{x,y\}) = \{x\}, \quad c(\{y,z\}) = \{y\}, \quad c(\{x,z\}) = \{z\}.$$

Each pairwise observation passes WARP, but together they imply $x \succ y \succ z \succ x$.

The content of a revealed-preference test depends on which feasible sets are observed.

Worked example: detecting a WARP violation

Two observations are

$$p^1 = (2, 1),\ x^1 = (4, 1), \qquad p^2 = (1, 2),\ x^2 = (1, 4).$$

Both observed expenditures equal 9. Cross-expenditures are

$$p^1 \cdot x^2 = 6 < 9, \qquad p^2 \cdot x^1 = 6 < 9.$$

Each chosen bundle was strictly affordable when the other bundle was selected. Thus x^1 is revealed preferred to x^2 and x^2 to x^1, violating WARP because the choices differ.

Application exercise: demand data

A consumer chooses

t	p^t	x^t
1	$(4, 1)$	$(3, 2)$
2	$(1, 4)$	$(2, 3)$
3	$(2, 2)$	$(2.5, 2.5)$

and exhausts the budget in each observation.

1. Construct the direct revealed-preference relation.
2. Test every pair of observations for WARP.
3. For each violating pair, compute the strict-affordability gaps in both observations.
4. Determine whether any rational, locally nonsatiated preference relation can rationalize the dataset.

Lecture 4: expected utility and risk aversion

Objective

Model choice under objective risk and connect curvature of Bernoulli utility to insurance, portfolio choice, and stochastic dominance.

Reading: MWG 6A–6C.

Lotteries over consequences

Let $C = \{c_1, \ldots, c_N\}$ be a finite set of consequences. A simple lottery is

$$L = (p_1, \ldots, p_N) \in \mathbb{R}^N_+, \qquad \sum_{n=1}^{N} p_n = 1.$$

The lottery assigns probability p_n to consequence c_n.

Examples

Consequences may be consumption bundles, monetary wealth levels, health outcomes, or asset returns.

Reduction of compound lotteries

Suppose lottery L_k occurs with probability α_k. The induced probability of consequence c_n is

$$p_n = \sum_{k=1}^{K} \alpha_k p_{nk}.$$

The consequentialist premise treats the compound lottery and this reduced simple lottery as equivalent.

Implication

Only final probabilities over consequences matter, not the sequence of randomizations that generates them.

The St. Petersburg paradox

Toss a fair coin until the first head. A head on toss n pays 2^n.

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} 2^{-n} 2^n = \infty.$$

Yet willingness to pay is finite for most decision makers.

Lesson

Expected monetary value does not describe choice when marginal utility of wealth decreases. Expected utility can assign a finite value to the gamble.

Expected utility over monetary risks

Let X be a random monetary payoff and let $u : \mathbb{R} \to \mathbb{R}$ be increasing and continuous. Expected utility is

$$U(X) = \mathbb{E}[u(X)] = \int u(x)\, dF_X(x).$$

The function u is the Bernoulli utility index. Under expected utility, the decision maker ranks risks by their expected Bernoulli utility.

Risk aversion and concavity

A decision maker is risk averse when

$$u(\mathbb{E}[X]) \geq \mathbb{E}[u(X)]$$

for every integrable risk X.

Jensen's inequality

Risk aversion is equivalent to concavity of u. Strict concavity implies strict preference for the mean over any nondegenerate risk with that mean.

Linear utility gives risk neutrality. Convex utility gives risk-loving behavior.

Certainty equivalent and risk premium

The certainty equivalent c_X solves

$$u(c_X) = \mathbb{E}[u(X)].$$

The risk premium is

$$\pi_X = \mathbb{E}[X] - c_X.$$

For a risk-averse decision maker, $c_X \leq \mathbb{E}[X]$ and $\pi_X \geq 0$.

Economic meaning

π_X is the largest amount the individual would pay to replace X with its expected value.

Arrow–Pratt absolute risk aversion

Let $X = w + \varepsilon$, with $\mathbb{E}[\varepsilon] = 0$ and variance σ^2. A second-order approximation gives

$$\pi_X \approx \frac{1}{2} A_u(w)\sigma^2, \qquad A_u(w) = -\frac{u''(w)}{u'(w)}.$$

$A_u(w)$ measures local aversion to additive risks. Concavity implies $A_u(w) \geq 0$ when $u' > 0$.

Arrow–Pratt equivalence theorem

Let u_1 and u_2 be increasing, twice differentiable utility functions. The following are equivalent under standard regularity conditions:

1. Decision maker 1 demands at least as large a risk premium for every risk than decision maker 2.
2. $u_1 = f \circ u_2$ for some increasing and concave f.
3. $A_{u_1}(w) \geq A_{u_2}(w)$ for every wealth level w.

Proof sketch I: curvature and concave transformations

Define $\phi = u_1 \circ u_2^{-1}$ on the range of u_2. Then $u_1 = \phi \circ u_2$ and, writing $z = u_2(w)$,

$$\phi'(z) = \frac{u_1'(w)}{u_2'(w)} > 0.$$

Differentiating once more gives

$$\phi''(z) = \frac{u_1'(w)}{[u_2'(w)]^2}\left(\frac{u_1''(w)}{u_1'(w)} - \frac{u_2''(w)}{u_2'(w)}\right) = \frac{u_1'(w)}{[u_2'(w)]^2}\big(A_{u_2}(w) - A_{u_1}(w)\big).$$

Hence ϕ is concave if and only if $A_{u_1}(w) \geq A_{u_2}(w)$ at every wealth level.

Proof sketch II: risk premia

Let c_i be individual i's certainty equivalent for a risk X: $u_i(c_i) = \mathbb{E}[u_i(X)]$. If $u_1 = \phi \circ u_2$ with ϕ increasing and concave, Jensen's inequality gives

$$\mathbb{E}[u_1(X)] = \mathbb{E}[\phi(u_2(X))] \leq \phi(\mathbb{E}[u_2(X)]) = \phi(u_2(c_2)) = u_1(c_2).$$

Since u_1 is increasing, $c_1 \leq c_2$. For a common mean μ,

$$\pi_1(X) = \mu - c_1 \geq \mu - c_2 = \pi_2(X).$$

Conversely, apply the assumed premium ordering to arbitrarily small symmetric risks around each w. The local expansion $\pi_i \simeq \frac{1}{2} A_{u_i}(w)\sigma^2$ yields $A_{u_1}(w) \geq A_{u_2}(w)$, which by the preceding slide makes u_1 an increasing concave transformation of u_2.

Common risk-aversion specifications

CARA $u(w) = -e^{-aw}$, $a > 0$, gives $A_u(w) = a$.

Log utility $u(w) = \log w$ gives $A_u(w) = 1/w$ and constant relative risk aversion equal to one.

CRRA $u(w) = w^{1-\gamma}/(1-\gamma)$ gives $A_u(w) = \gamma/w$ and relative risk aversion γ.

Log and CRRA utility exhibit decreasing absolute risk aversion.

Relative risk aversion

For proportional risk $X = w(1+\varepsilon)$, define

$$R_u(w) = -\frac{wu''(w)}{u'(w)} = wA_u(w).$$

If $\mathbb{E}[\varepsilon] = 0$ and its variance is σ^2, the proportional risk premium τ satisfies approximately

$$\tau \approx \frac{1}{2} R_u(w)\sigma^2.$$

Relative risk aversion measures local aversion to risks that scale with wealth.

First-order stochastic dominance

X first-order stochastically dominates Y when every increasing expected-utility decision maker weakly prefers X:

$$\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)] \quad \text{for every increasing } u.$$

Equivalently,

$$F_X(z) \leq F_Y(z) \quad \text{for every } z.$$

At every threshold, X assigns weakly less probability to outcomes below that threshold.

Second-order stochastic dominance

X second-order stochastically dominates Y when every increasing, concave expected-utility decision maker weakly prefers X. For risks with the same mean, this is equivalent to

$$\int_{-\infty}^{z} F_X(t)\,dt \le \int_{-\infty}^{z} F_Y(t)\,dt \quad \text{for every } z.$$

Y is a mean-preserving spread of X: it adds risk without changing the mean.

Worked example: demand for insurance

Initial wealth is w. A loss D occurs with probability q. Coverage I costs πI and pays I in the loss state.

$$EU(I) = (1-q)u(w-\pi I) + q\,u(w-D+(1-\pi)I).$$

The interior first-order condition is

$$(1-q)\pi u'(w-\pi I) = q(1-\pi)u'(w-D+(1-\pi)I).$$

With actuarially fair pricing $\pi = q$ and strictly concave utility, full insurance $I = D$ equalizes wealth across states.

Application exercise: portfolio choice

An investor with wealth w places amount a in a risky asset with excess return $\widetilde{R}$ and keeps the rest in a safe asset. Final wealth is $w + a\widetilde{R}$.

1. Derive the first- and second-order conditions for optimal a.
2. For a small risk, obtain an approximation to a^* using $A_u(w)$, $\mathbb{E}[\widetilde{R}]$, and $\mathrm{Var}(\widetilde{R})$.
3. Compare optimal risky investment for two agents ordered by Arrow–Pratt risk aversion.
4. Rework the approximation for a proportional position and interpret relative risk aversion.

Lecture 5: expected utility representation

Objective

Identify the axioms that yield a von Neumann–Morgenstern representation and examine prominent violations.

Reading: MWG 6A–6B.

Expected utility representation

A preference relation on the lottery simplex $\mathcal{L}$ has an expected utility representation if some $v : C \to \mathbb{R}$ satisfies

$$U(L) = \sum_{n=1}^{N} p_n v(c_n)$$

and

$$L \succeq L' \iff U(L) \geq U(L').$$

The representation is linear in probabilities, not necessarily linear in consequences.

Continuity on the lottery simplex

For lotteries L, L', L'', continuity requires the sets

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succeq L''\}$$

and

$$\{\alpha \in [0,1] : L'' \succeq \alpha L + (1-\alpha)L'\}$$

to be closed. Equivalently, if $L \succeq L' \succeq L''$, there exists $\alpha \in [0,1]$ such that $L' \sim \alpha L + (1-\alpha)L''$.

The independence axiom

For any $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$,

$$L \succeq L' \iff \alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''.$$

Mixing both lotteries with the same third lottery does not change their ranking.

Role in the theorem

Independence creates the linearity in probabilities that characterizes expected utility.

von Neumann–Morgenstern theorem

Representation theorem

A rational preference relation on $\mathcal{L}$ that satisfies continuity and independence admits an expected utility representation.

The Bernoulli index v is unique up to a positive affine transformation:

$$\widetilde{v}(c) = a + bv(c), \qquad b > 0.$$

General increasing transformations do not preserve the expected utility form.

Proof sketch I: canonical lotteries

Because the consequence set is finite, choose a best consequence c_B and a worst consequence c_W. Assume $c_B \succ c_W$.

Continuity step

For each consequence c_n, continuity gives a number $\alpha_n \in [0, 1]$ such that

$$c_n \sim \alpha_n c_B + (1 - \alpha_n) c_W.$$

Independence implies that canonical lotteries are ordered by their probability of the best outcome:

$$\alpha c_B + (1 - \alpha) c_W \succeq \beta c_B + (1 - \beta) c_W \iff \alpha \geq \beta.$$

Hence each α_n is unique.

Proof sketch II: replace consequences by lotteries

Let $L = (p_1, \dots, p_N)$. Independence preserves indifference after mixing with a common lottery. Therefore each occurrence of c_n can be replaced by its canonical equivalent:

$$L = \sum_{n=1}^{N} p_n c_n \sim \sum_{n=1}^{N} p_n \big[\alpha_n c_B + (1-\alpha_n) c_W\big].$$

Reducing the compound lottery gives

$$L \sim \left(\sum_{n=1}^{N} p_n \alpha_n\right) c_B + \left(1 - \sum_{n=1}^{N} p_n \alpha_n\right) c_W.$$

Every lottery is therefore equivalent to one canonical best–worst lottery.

Proof sketch III: the expected utility index

Define $v(c_n) = \alpha_n$ and

$$U(L) = \sum_{n=1}^{N} p_n v(c_n) = \sum_{n=1}^{N} p_n \alpha_n.$$

The previous step gives

$$L \sim U(L) c_B + [1 - U(L)] c_W.$$

Since canonical lotteries are ranked by the probability of c_B,

$$L \succeq L' \iff U(L) \geq U(L').$$

Thus U represents preferences and is linear in probabilities, which is the expected utility form.

Proof sketch IV: affine uniqueness

Let $\widetilde{v}$ be another Bernoulli index with an expected utility representation. Normalize both indices at the extremes:

$$v(c_W) = \widetilde{v}(c_W) = 0, \qquad v(c_B) = \widetilde{v}(c_B) = 1.$$

The indifference

$$c_n \sim \alpha_n c_B + (1 - \alpha_n) c_W$$

forces $v(c_n) = \widetilde{v}(c_n) = \alpha_n$ for every n. The normalized indices coincide. Undoing the normalization gives

$$\widetilde{v}(c) = a + bv(c), \qquad b > 0.$$

Positive affine transformations are therefore the only transformations that preserve the expected utility form.

The Allais choices

Let outcomes be $(\text{EUR } 2.5\text{m}, \text{EUR } 0.5\text{m}, 0)$ and consider

	EUR 2.5m	EUR 0.5m	0
L_1	0	1	0
L_1'	0.10	0.89	0.01
L_2	0	0.11	0.89
L_2'	0.10	0	0.90

A common pattern is $L_1 \succ L_1'$ together with $L_2' \succ L_2$.

Why the Allais pattern violates expected utility

Expected utility implies

$$L_1 \succ L_1' \implies 0.11v(0.5) > 0.10v(2.5) + 0.01v(0),$$

while

$$L_2' \succ L_2 \implies 0.10v(2.5) + 0.01v(0) > 0.11v(0.5).$$

The inequalities contradict each other. The observed certainty effect therefore violates independence.

Regret and the Machina example

Consider three outcomes: a trip to Venice, a film about Venice, and staying home.

- A small chance of staying home may be preferable to an equal chance of watching the film after missing the trip.
- The final outcome may feel different because it is evaluated against the foregone alternative.
- Expected utility over final consequences excludes this form of regret by construction.

Worked example: constructing a vNM index

Let $c_B \succ c \succ c_W$ denote best, intermediate, and worst consequences. Normalize

$$v(c_B) = 1, \qquad v(c_W) = 0.$$

Continuity gives a probability $\alpha \in (0, 1)$ such that

$$c \sim \alpha c_B + (1 - \alpha) c_W.$$

Set $v(c) = \alpha$. Repeating this calibration for every consequence constructs a Bernoulli index. Independence then extends the ranking linearly to all lotteries.

Application exercise: axioms and calibration

Outcomes satisfy $c_3 \succ c_2 \succ c_1$. A decision maker reports

$$c_2 \sim \tfrac{3}{4}c_3 + \tfrac{1}{4}c_1.$$

1. Normalize $v(c_3) = 1$ and $v(c_1) = 0$. Recover $v(c_2)$.
2. Rank $L = (0.4, 0.4, 0.2)$ and $L' = (0.5, 0.2, 0.3)$ under expected utility, where probabilities correspond to (c_3, c_2, c_1).
3. Construct an independence test using a common mixture with c_1.
4. Give a preference reversal that preserves continuity and rationality but violates independence.