

INDIVIDUAL DECISION MAKING

Mathematical Appendix *

Masters M1 MAEF

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1 Notations

- $\mathbb{R}^n := \{x = (x^1, \dots, x^h, \dots, x^n) : x^h \in \mathbb{R}, \forall h = 1, \dots, n\}$
- $x \in \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$,

$$x \geq \bar{x} \iff x^h \geq \bar{x}^h, \forall h = 1, \dots, n$$

$$x > \bar{x} \iff x \geq \bar{x} \text{ and } x \neq \bar{x}$$

$$x \gg \bar{x} \iff x^h > \bar{x}^h, \forall h = 1, \dots, n$$

- $x \in \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$, $x \cdot \bar{x}$ denotes the scalar product of x and $\bar{x}$.
- A is a matrix with m rows and n columns and B is a matrix with n rows and l columns, AB denotes the matrix product of A and B .
- H is a $n \times n$ matrix, $\text{tr}(H)$ denotes the trace of H and $\det(H)$ denotes the determinant of H .
- $x \in \mathbb{R}^n$ is treated as a row matrix.
- x^T denotes the transpose of $x \in \mathbb{R}^n$, x^T is treated as a column matrix.
- f is a function from $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$,

f is **weakly increasing (or non-decreasing)** on X if for all x and $\bar{x}$ in X ,

$$x \leq \bar{x} \implies f(x) \leq f(\bar{x})$$

f is **increasing** on X if for all x and $\bar{x}$ in X ,

$$x \ll \bar{x} \implies f(x) < f(\bar{x})$$

f is **strictly increasing** on X if for all x and $\bar{x}$ in X ,

$$x < \bar{x} \implies f(x) < f(\bar{x})$$

f strictly increasing on $X \implies f$ increasing on X

f strictly increasing on $X \implies f$ weakly increasing (or non-decreasing) on X

- $X \subseteq \mathbb{R}^n$ is an open set, f is a function from X to $\mathbb{R}$ and $x \in X$,

$$\nabla f(x) := \left(\frac{\partial f}{\partial x^1}(x), \dots, \frac{\partial f}{\partial x^h}(x), \dots, \frac{\partial f}{\partial x^n}(x) \right)$$

denotes the **gradient** of f at x , and

$$\mathbf{H}f(x) := \begin{bmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1}(x) & \dots & \frac{\partial^2 f}{\partial x^h \partial x^1}(x) & \dots & \frac{\partial^2 f}{\partial x^n \partial x^1}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x^1 \partial x^h}(x) & \dots & \frac{\partial^2 f}{\partial x^h \partial x^h}(x) & \dots & \frac{\partial^2 f}{\partial x^n \partial x^h}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x^1 \partial x^n}(x) & \dots & \frac{\partial^2 f}{\partial x^h \partial x^n}(x) & \dots & \frac{\partial^2 f}{\partial x^n \partial x^n}(x) \end{bmatrix}_{n \times n}$$

denotes the **Hessian matrix** of f at x .

- $X \subseteq \mathbb{R}^n$ is an open set, $g := (g_1, \dots, g_j, \dots, g_m)$ is a mapping from X to $\mathbb{R}^m$ and $x \in X$,

$$\mathbf{J}g(x) := \begin{bmatrix} \frac{\partial g_1}{\partial x^1}(x) & \dots & \frac{\partial g_1}{\partial x^h}(x) & \dots & \frac{\partial g_1}{\partial x^n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_j}{\partial x^1}(x) & \dots & \frac{\partial g_j}{\partial x^h}(x) & \dots & \frac{\partial g_j}{\partial x^n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_m}{\partial x^1}(x) & \dots & \frac{\partial g_m}{\partial x^h}(x) & \dots & \frac{\partial g_m}{\partial x^n}(x) \end{bmatrix}_{m \times n} = \begin{bmatrix} \nabla g_1(x) \\ \vdots \\ \nabla g_j(x) \\ \vdots \\ \nabla g_m(x) \end{bmatrix}_{m \times n}$$

denotes the **Jacobian matrix** of g at x .

1.1 Continuity

f is a function from $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$.

Definition 1 (Continuous function) f is continuous at $\bar{x} \in X$ if

$$\lim_{x \rightarrow \bar{x}} f(x) = f(\bar{x})$$

f is continuous on X if f is continuous at every point $\bar{x} \in X$.

Exercise 2

1. f is continuous at $\bar{x} \in X$ if and only if for every open ball J of center $f(\bar{x})$ there exists an open ball B of center $\bar{x}$ such that $f(B \cap X) \subseteq J$.
2. f is continuous at $\bar{x} \in X$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - \bar{x}\| < \delta$ and $x \in X \implies |f(x) - f(\bar{x})| < \varepsilon$.

Proposition 3 (Sequentially continuous function) f is continuous at $\bar{x} \in X$ if and only if f is sequentially continuous at $\bar{x}$, that is, for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ such that $x_n \rightarrow \bar{x}$, we have that

$$f(x_n) \rightarrow f(\bar{x})$$

1.2 Differentiability

$X \subseteq \mathbb{R}^n$ is an **open** set, f is a function from X to $\mathbb{R}$.

Definition 4 (Differentiable function) f is differentiable at $\bar{x} \in X$ if

1. all the partial derivatives of f at $\bar{x}$ exist,
2. there exists a function $E_{\bar{x}}$ defined in some open ball $B(0, \varepsilon) \subseteq \mathbb{R}^n$ such that for every $u \in B(0, \varepsilon)$,

$$f(\bar{x} + u) = f(\bar{x}) + \nabla f(\bar{x}) \cdot u + \|u\| E_{\bar{x}}(u)$$

$$\text{where } \lim_{u \rightarrow 0} E_{\bar{x}}(u) = 0$$

f is differentiable on X if f is differentiable at every point $\bar{x} \in X$.

Exercise 5 If f is differentiable at $\bar{x}$, then f is continuous at $\bar{x}$.

Definition 6 (Directional derivative) Let $v \in \mathbb{R}^n$, $v \neq 0$. The directional derivative $D_v f(\bar{x})$ of f at $\bar{x} \in X$ in the direction v with $\|v\| = 1$ is defined as:

$$\lim_{t \rightarrow 0^+} \frac{f(\bar{x} + tv) - f(\bar{x})}{t}$$

if this limit exists and it is finite.

Proposition 7 (Differentiable function/Directional derivative) If f is differentiable at $\bar{x} \in X$, then for every $v \in \mathbb{R}^n$ with $v \neq 0$,

$$D_v f(\bar{x}) = \nabla f(\bar{x}) \cdot v$$

1.3 Compactness

X is a subset of $\mathbb{R}^n$.

Proposition 8 (Compact set/Subsequences) X is compact if and only if for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of the sequence $(x_n)_{n \in \mathbb{N}}$ such that $(x_{n_k})_{k \in \mathbb{N}}$ converges to some point $\bar{x} \in X$.¹

Proposition 9 (Compact set) X is compact if and only if it is closed and bounded.

Definition 10 (Closed set) X is closed if its complement $\mathcal{C}(X) := \mathbb{R}^n \setminus X$ is open.

Proposition 11 (Sequentially closed) X is closed if and only if it is sequentially closed, that is, for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ such that $x_n \rightarrow \bar{x}$, we have

$$\bar{x} \in X$$

Definition 12 (Bounded set) X is bounded if it is included in some ball, that is, there exists $\varepsilon > 0$ such that for all $x \in X$, $\|x\| < \varepsilon$.

¹Let $(x_n)_{n \in \mathbb{N}}$ be a sequence and $(n_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers. The composed sequence $(x_{n_k})_{k \in \mathbb{N}}$ is a subsequence of the sequence $(x_n)_{n \in \mathbb{N}}$.

2 Extreme Value Theorem

Theorem 13 (Extreme Value Theorem/Weierstrass Theorem) *Let f be a function from $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$. If X is a non-empty compact set and f is continuous on X , then*

- $\exists x^* \in X$ such that $f(x^*) \geq f(x)$ for all $x \in X$, and
- $\exists x^{**} \in X$ such that $f(x^{**}) \leq f(x)$ for all $x \in X$.

3 Karush–Kuhn–Tucker Conditions

In this section, we focus on necessary and sufficient conditions in terms of first-order conditions for solving a maximization problem with inequality constraints.

In this section, we assume that

- $C \subseteq \mathbb{R}^n$ is **convex and open**,
- the following functions f and g_j with $j = 1, \dots, m$ are **differentiable** on C .

$$\begin{aligned} f : x \in C \subseteq \mathbb{R}^n &\longrightarrow f(x) \in \mathbb{R} \text{ and} \\ g_j : x \in C \subseteq \mathbb{R}^n &\longrightarrow g_j(x) \in \mathbb{R}, \forall j = 1, \dots, m \end{aligned}$$

Maximization problem

$$\begin{aligned} \max_{x \in C} \quad & f(x) \\ \text{subject to} \quad & g_j(x) \geq 0, \forall j = 1, \dots, m \end{aligned} \tag{1}$$

where f is the *objective* function, and g_j with $j = 1, \dots, m$ are the *constraint* functions.

The **Karush–Kuhn–Tucker conditions** associated with problem (1) are given below

$$\left\{ \begin{aligned} \nabla f(x) + \sum_{j=1}^m \lambda_j \nabla g_j(x) &= 0 \\ \lambda_j &\geq 0, \forall j = 1, \dots, m \\ \lambda_j g_j(x) &= 0, \forall j = 1, \dots, m \\ g_j(x) &\geq 0, \forall j = 1, \dots, m \end{aligned} \right. \tag{2}$$

where for every $j = 1, \dots, m$, $\lambda_j \in \mathbb{R}$ is called *Lagrange multiplier* associated with the inequality constraint g_j .

Definition 14 Let $x^* \in C$, we say that the constraint j is **binding** at x^* if $g_j(x^*) = 0$. We denote

1. $B(x^*)$ the set of all binding constraints at x^* , that is

$$B(x^*) := \{j = 1, \dots, m : g_j(x^*) = 0\}$$

2. $m^* \leq m$ the number of elements of $B(x^*)$ and

3. $g^* := (g_j)_{j \in B(x^*)}$ the following mapping

$$g^* : x \in C \subseteq \mathbb{R}^n \longrightarrow g^*(x) = (g_j(x))_{j \in B(x^*)} \in \mathbb{R}^{m^*}$$

Theorem 15 (Karush–Kuhn–Tucker are necessary conditions) *Let x^* be a solution to problem (1). Assume that **one** of the following conditions is satisfied.*

1. *For all $j = 1, \dots, m$, g_j is a **linear or affine** function.*

2. **Slater's Condition :**

- *for all $j = 1, \dots, m$, g_j is a **concave** function **or** g_j is a **quasi-concave** function with $\nabla g_j(x) \neq 0$ for all $x \in C$, and*
- *there exists $\bar{x} \in C$ such that $g_j(\bar{x}) > 0$ for all $j = 1, \dots, m$.*

3. **Rank Condition :** $\text{rank } Jg^*(x^*) = m^* \leq n$.

Then, there exists $\lambda^ = (\lambda_1^*, \dots, \lambda_j^*, \dots, \lambda_m^*) \in \mathbb{R}_+^m$ such that (x^*, λ^*) satisfies the Karush–Kuhn–Tucker Conditions (2).*

Theorem 16 (Karush–Kuhn–Tucker are sufficient conditions) *Suppose that there exists $\lambda^* = (\lambda_1^*, \dots, \lambda_j^*, \dots, \lambda_m^*) \in \mathbb{R}_+^m$ such that $(x^*, \lambda^*) \in C \times \mathbb{R}_+^m$ satisfies the Karush–Kuhn–Tucker Conditions (2). Assume that*

1. *f is a **concave** function **or** f is a **quasi-concave** function with $\nabla f(x) \neq 0$ for all $x \in C$, and*
2. *g_j is a **quasi-concave** function for all $j = 1, \dots, m$.*

Then, x^ is a solution to problem (1).*

4 Concavity and quasi-concavity

In this section, we assume that C is a **convex** subset of $\mathbb{R}^n$ and f is a function from C to $\mathbb{R}$.

Concavity

Definition 17 (Concave function) f is concave if for all $t \in [0, 1]$ and for all x and $\bar{x}$ in C ,

$$f(tx + (1 - t)\bar{x}) \geq tf(x) + (1 - t)f(\bar{x})$$

Proposition 18 f is concave **if and only if** the set

$$\{(x, \alpha) \in C \times \mathbb{R} : f(x) \geq \alpha\}$$

is a convex subset of $\mathbb{R}^{n+1}$. The set above is called hypograph of f .

Proposition 19 C is **open** and f is **differentiable** on C . f is concave **if and only if** for all x and $\bar{x}$ in C ,

$$f(x) \leq f(\bar{x}) + \nabla f(\bar{x}) \cdot (x - \bar{x})$$

Proposition 20 C is **open** and f is **twice continuously differentiable** on C . f is concave **if and only if** for all $x \in C$ the Hessian matrix $Hf(x)$ is negative semidefinite, that is, for all $x \in C$

$$vHf(x)v^T \leq 0, \forall v \in \mathbb{R}^n$$

Definition 21 (Strictly concave function) f is strictly concave if for all $t \in]0, 1[$ and for all x and $\bar{x}$ in C with $x \neq \bar{x}$,

$$f(tx + (1 - t)\bar{x}) > tf(x) + (1 - t)f(\bar{x})$$

Proposition 22 C is **open** and f is **differentiable** on C . f is strictly concave **if and only if** for all x and $\bar{x}$ in C with $x \neq \bar{x}$,

$$f(x) < f(\bar{x}) + \nabla f(\bar{x}) \cdot (x - \bar{x})$$

Proposition 23 C is **open** and f is **twice continuously differentiable** on C . **If** for all $x \in C$ the Hessian matrix $Hf(x)$ is negative definite, that is, for all $x \in C$

$$vHf(x)v^T < 0, \forall v \in \mathbb{R}^n, v \neq 0$$

then f is strictly concave.

Quasi-concavity

Definition 24 (Quasi-concave function) f is quasi-concave if and only if for all $\alpha \in \mathbb{R}$ the set

$$\{x \in C : f(x) \geq \alpha\}$$

is a convex subset of $\mathbb{R}^n$. The set above is called upper contour set of f at α .

Proposition 25 f is quasi-concave **if and only if** for all $t \in [0, 1]$ and for all x and $\bar{x}$ in C ,

$$f(tx + (1 - t)\bar{x}) \geq \min\{f(x), f(\bar{x})\}$$

Proposition 26 C is **open** and f is **differentiable** on C . f is quasi-concave **if and only if** for all x and $\bar{x}$ in C ,

$$f(x) \geq f(\bar{x}) \implies \nabla f(\bar{x}) \cdot (x - \bar{x}) \geq 0$$

Proposition 27 C is **open** and f is **differentiable** on C . **If** f is quasi-concave and $\nabla f(x) \neq 0$ for all $x \in C$, **then** for all x and $\bar{x}$ in C with $x \neq \bar{x}$,

$$f(x) > f(\bar{x}) \implies \nabla f(\bar{x}) \cdot (x - \bar{x}) > 0$$

Proposition 28 C is **open** and f is **twice continuously differentiable** on C . **If** f is quasi-concave, **then** for all $x \in C$ the Hessian matrix $Hf(x)$ is negative semidefinite on $\text{Ker } \nabla f(x)$, that is, for all $x \in C$

$$v \in \mathbb{R}^n \text{ and } \nabla f(x) \cdot v = 0 \implies vHf(x)v^T \leq 0$$

Definition 29 (Strictly quasi-concave function) f is strictly quasi-concave if and only if for all $t \in]0, 1[$ and for all x and $\bar{x}$ in C with $x \neq \bar{x}$,

$$f(tx + (1 - t)\bar{x}) > \min\{f(x), f(\bar{x})\}$$

Proposition 30 C is **open** and f is **differentiable** on C .

1. **If** for all x and $\bar{x}$ in C with $x \neq \bar{x}$,

$$f(x) \geq f(\bar{x}) \implies \nabla f(\bar{x}) \cdot (x - \bar{x}) > 0$$

then f is strictly quasi-concave.

2. **If** f is strictly quasi-concave and $\nabla f(x) \neq 0$ for all $x \in C$, **then** for all x and $\bar{x}$ in C with $x \neq \bar{x}$,

$$f(x) \geq f(\bar{x}) \implies \nabla f(\bar{x}) \cdot (x - \bar{x}) > 0$$

Proposition 31 C is **open** and f is **twice continuously differentiable** on C . **If** for all $x \in C$ the Hessian matrix $Hf(x)$ is negative definite on $\text{Ker} \nabla f(x)$, that is, for all $x \in C$

$$v \in \mathbb{R}^n, v \neq 0 \text{ and } \nabla f(x) \cdot v = 0 \implies vHf(x)v^T < 0$$

then f is strictly quasi-concave.

Remark 32 We remark that

$$\begin{array}{ccccc} f \text{ linear or affine} & \Rightarrow & f \text{ concave} & \Leftarrow & f \text{ strictly concave} \\ & & \Downarrow & & \Downarrow \\ & & f \text{ quasi-concave} & \Leftarrow & f \text{ strictly quasi-concave} \end{array}$$

We remind the definitions and some properties of negative definite/semidefinite matrices. Let H be a $n \times n$ **symmetric** matrix.

Definition 33

1. H is negative semidefinite if $vHv^T \leq 0$ for all $v \in \mathbb{R}^n$.
2. H is negative definite if $vHv^T < 0$ for all $v \in \mathbb{R}^n$ with $v \neq 0$.

Proposition 34

1. H has n real eigenvalues. We denote $\lambda_1, \dots, \lambda_n$ the eigenvalues of H .
2. H is negative semidefinite if and only $\lambda_i \leq 0$ for every $i = 1, \dots, n$.
3. H is negative definite if and only $\lambda_i < 0$ for every $i = 1, \dots, n$.

Proposition 35

1. If H is negative semidefinite, then $\text{tr}(H) \leq 0$ and $\det(H) \geq 0$ if n is even, $\det(H) \leq 0$ if n is odd.

2. If H is negative definite, then $\text{tr}(H) < 0$ and $\det(H) > 0$ if n is even, $\det(H) < 0$ if n is odd.

We remark that if $n = 2$, then the conditions stated in the proposition above also are sufficient conditions, that is

1. H is negative semidefinite if and only if $\text{tr}(H) \leq 0$ and $\det(H) \geq 0$.
2. H is negative definite if and only if $\text{tr}(H) < 0$ and $\det(H) > 0$.

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