C1. Suppose that $\succsim$ is rational. Prove that if $x \succ y \succsim z$, then $x \succ z$.

A/: Since $\succsim$ is rational, it is complete. Therefore, ether $x \succsim z$ or $z \succsim x$. Since $x \succ y$ by hypothesis, then $x \succsim y$. By transitivity, given that $y \succsim z$, we have that $\boxed{x \succsim z}$ (Transitivity is provided since $\succsim$ is rational)

Now: $x \succ y \Rightarrow \neg(y \succsim x)$ $\quad$ Aim: $\neg(z \succsim x)$

By contradiction, suppose $z \succsim x$. Since $x \succ y$, by transitivity: $z \succsim y$ the latter implies, since by hypothesis $y \succsim z$, that $z \sim y$, which in turn implies $y \succsim x$, because we assumed $z \succsim x$. But this is a contradiction, because $x \succ y$.

C2. Suppose that $\succsim$ is rational. Prove the following:

a) $\succ$ is both reflexive ($x \succ x$ never holds) and transitive

A/ $\succ$ is reflexive: supp. $\exists x : x \succ x \rightarrow x \succsim x$ and $\neg(x \succsim x)$ ↯

$\succ$ is transitive: supp $x \succ y$ and $y \succ z$ $\quad$ Aim: $x \succ z$

since $y \succ z$: $y \succsim z$. By c1 $x \succ z$.

b) $\sim$ is reflexive ($x \sim x \ \forall x$), transitive and symmetric ($x \sim y \Rightarrow y \sim x$)

A/ $\sim$ is reflexive: since $\succsim$ is complete (it is rational):

$x \succsim x$ or $x \succsim x \rightarrow x \sim x$

$\sim$ is transitive: supp. $x \sim y$ and $y \sim z$. Since $\succsim$ is transitive:

$x \sim y \rightarrow x \succsim y$ and $y \sim z \rightarrow y \succsim z \Rightarrow x \succsim z$. The same reasoning applies, by the reflexiveness of $\sim$: $y \sim x \rightarrow y \succsim x$ and $z \sim y \rightarrow z \succsim y \Rightarrow z \succsim x$. Then: $x \sim z$

$\sim$ is symmetric: supp. $x \sim y$ then $x \succsim y$ and $y \succsim x \Rightarrow y \sim x$.

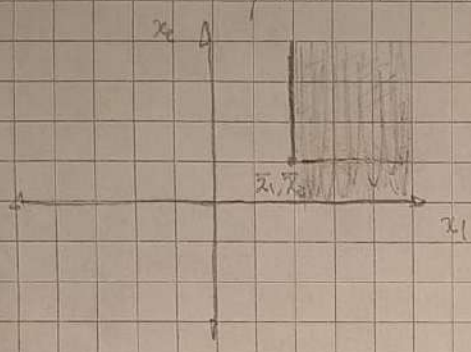
C6. Lexicographic preferences. $\forall\, x=(x_1,x_2) \in \mathbb{R}^2_+$ and $\bar{x}=(\bar{x}_1,\bar{x}_2) \in \mathbb{R}^2_+$:

$$x \succsim \bar{x} \iff x_1 > \bar{x}_1 \quad \text{or} \quad x_1 = \bar{x}_1 \ \& \ x_2 \geq \bar{x}_2$$

a) $\forall\, \bar{x} \in \mathbb{R}^2_+$, determine and draw the upper contour set $U(\bar{x})$.

A: Let $\bar{x}=(\bar{x}_1,\bar{x}_2) \in \mathbb{R}^2$

$$x \in U(\bar{x}) : \{ x \in \mathbb{R}^2_+ : x \succsim \bar{x} \}$$



b) Show that $\forall\, \bar{x} \in \mathbb{R}^2_+$, the indifference set is a singleton.

A: First find $I(\bar{x}) = \{ x \in \mathbb{R}^2_+ : x \sim \bar{x} \}$

If x s.t. (x_1, x_2)

$x \succsim \bar{x} \rightarrow$ either $x_1 > \bar{x}_1$ or $x_1 = \bar{x}_1$ & $x_2 \geq \bar{x}_2$

and

$\bar{x} \succsim x \rightarrow$ either $x_1 < \bar{x}_1$ or $\bar{x}_1 = x_1$ & $\bar{x}_2 \geq x_2$

First: $x_1 \not> \bar{x}_1$ and $\bar{x}_1 \not> x_1 \rightarrow x_1 = \bar{x}_1$

Now: $x_2 \geq \bar{x}_2$ and $\bar{x}_2 \geq x_2 \rightarrow x_2 = \bar{x}_2$

Conclusion: $x_1 = \bar{x}_1$ & $x_2 = \bar{x}_2 \Rightarrow x = \{\bar{x}\} = I(\bar{x})$

c) Show that this preference is strongly monotone and strictly convex, but not continuous.

A: St monotonicity: Suppose $x=(x_1,x_2) > \bar{x}=(\bar{x}_1,\bar{x}_2)$. Then:

$x_1 > \bar{x}_1$ & $x_2 > \bar{x}_2 \underset{\text{By def}}{\Rightarrow} x \succsim \bar{x}$

Now: $\bar{x} \not\succsim x$ Because $\bar{x}_1 < x_1$

Strictly convex: let $y=(y_1,y_2)$, $z=(z_1,z_2)$, $x=(x_1,x_2)$, with $y \neq z$, such that $y \succsim x$ and $z \succsim x$.

since $y \succsim x$: $y_1 > x_1$ or $y_1 = x_1$, $y_2 \geq x_2$

since $z \succsim x$: $z_1 > x_1$ or $z_1 = x_1$, $z_2 \geq x_2$

Lim: $\lambda y + (1-\lambda) z \succsim x$, $\forall \lambda \in [0,1]$

so we have $\lambda y_1 + (1-\lambda) z_1 > x_1$ or $\lambda y_1 + (1-\lambda) z_1 = x_1$ and $\lambda y_2 + (1-\lambda) z_2 > x_2$

in both cases: $\lambda y + (1-\lambda) z \succ x$.

Not continuous:

Lim: find seq. (x_n) and (y_n) s.t. $x_n \succsim y_n$ $\forall n$, but $\lim_{n\to\infty} x_n \not\succsim \lim_{n\to\infty} y_n$

Let $y_n = (1,0)$, $\forall n$, then $\lim_{n\to\infty} y_n = (1,0) = y$

and $x_n = (1-1/n, 1)$. Then: $y_n \succsim x_n$, but: $\lim_{n\to\infty} x_n = (1,1) \succ y$

It is concluded that $\succsim$ is not continuous

Slide 24. A household chooses heating fuel x_1 and electricity x_2.
Technical constraints make the services complementary. Define

$$x \succsim y \iff \min\{x_1, 2x_2\} \geq \min\{y_1, 2y_2\}$$

1. Prove that $\succsim$ is rational and continuous

A/ Let $x, y \in \mathbb{R}^2_+$ (Completeness)

$x = (x_1, x_2)$ let $x_- = \min\{x_1, 2x_2\} \in \mathbb{R}_+$

$y = (y_1, y_2)$ and $y_- = \min\{y_1, 2y_2\} \in \mathbb{R}_+$

Min: either $x \succsim y$ or $y \succsim x$

Since $\mathbb{R}$ is complete: either $x_- \geq y_-$ or $y_- \geq x_-$

then $\succsim$ is complete

(Transitivity) Consider x, y, z such that $x \succsim y$, $y \succsim z$ Min $x \succsim z$

since $x \succsim y$ $x_- \geq y_-$

since $y \succsim z$ $y_- \geq z_-$ $\Rightarrow x_- \geq z_- \Rightarrow$ since transitivity is satisfied in $\mathbb{R}^+$

$x \succsim z$.

(Continuity). let $(x_n), (y_n)$ s.t. $\forall n$. $x_n \succsim y_n$ and $x_n \to x$, $y_n \to y$

Min: $x \succsim y$

Note that $\min(\cdot)$, $x \mapsto x_1$ and $x \mapsto 2x_2$ are continuous functions

Then $\min\{x_1, 2x_2\}$ is continuous

Then $\lim_{n\to\infty} \min\{x_1^n, 2x_2^n\} \to \min\{x_1, 2x_2\}$

and $\lim_{n\to\infty} \min\{y_1^n, 2y_2^n\} \to \min\{y_1, 2y_2\}$

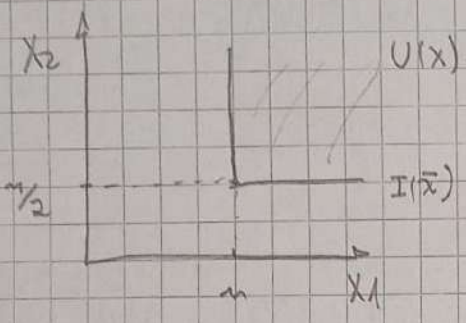
since $x^n \succsim y^n$ $\forall n$ $\min\{x_1, 2x_2\} \geq \min\{y_1, 2y_2\} \Rightarrow x \succsim y$

2. Characterize $U(\bar{x})$ and $I(\bar{x})$ for an arbitrary $\bar{x} \in \mathbb{R}^2_+$.

A: Let $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2_+$.

Supp. $x \in U(\bar{x})$: $x \succsim \bar{x} \iff \min\{x_1, 2x_2\} \geq \min\{\bar{x}_1, 2\bar{x}_2\}$

Let $\min\{\bar{x}_1, 2\bar{x}_2\} = m$. Then $U(\bar{x}) = \{(x_1, x_2) : \min\{x_1, x_2\} \geq m\}$.



whenever $x_1 \geq m$: $x_2 \geq m/2$

whenever $2x_2 \geq m$: $x_1 \geq m$

3. Determine whether:

- $\succsim$ is monotone, strongly monotone.

A: supp $y > x$.

Aim: $y \succsim x$

Then $y_1 > x_1$ and $y_2 > x_2 \Rightarrow 2y_2 > 2x_2$

and therefore: $\min\{y_1, 2y_2\} > \min\{x_1, 2x_2\}$.

But not strictly:

consider $x = (1, 1/2)$
$y = (2, 1/2)$

then $y \succsim x$. Indeed $\min\{y_1, 2y_2\} = 1 = \min\{x_1, 2x_2\}$

But $x \succsim y$ for the same reason

then $y \not\succ x$ which implies that $\succsim$ is not strictly monotone

- $\succsim$ is locally non sacated.

Yes. It follows from the statement: monotonicity $\Rightarrow$ local non sacation

∴ Convex

Consider x, y, z s.t: $y \succsim x$ and $z \succsim x$ $\quad \boxed{\text{Aim: } \lambda y + (1-\lambda) z \succsim x}$

Lema. Let $g_1(y) = y_1$, $g_2(y) = 2y_2$.

Note that $\lambda y + (1-\lambda) z = (\lambda y_1 + (1-\lambda) z_1, \lambda y_2 + (1-\lambda) z_2)$

$\min \{ g_i(\lambda y + (1-\lambda) z), g_2(\lambda y + (1-\lambda) z) \}$ (a)

$= \boxed{\min \{ \lambda y_1 + (1-\lambda) z_1, 2[\lambda y_2 + (1-\lambda) z_2] \} \geq \lambda \min \{ y_1, 2y_2 \} + (1-\lambda) \min \{ z_1, 2z_2 \}}$

① $g_i(y) \geq \min \{ g_1(y), g_2(y) \}$
$g_i(z) \geq \min \{ g_1(z), g_2(z) \}$
$\Rightarrow \lambda g_i(y) + (1-\lambda) g_i(z) \geq \lambda \min \{ g_1(y), g_2(y) \} + (1-\lambda) \min \{ g_1(z), g_2(z) \}$

Because g_i is linear

$\Rightarrow g_i(\lambda y + (1-\lambda) z) \geq \lambda \min \{ g_1(y), g_2(y) \} + (1-\lambda) \min \{ g_1(z), g_2(z) \}$

For both $i = 1$ and 2. Then, in particular, for $\min \{ g_1(\cdot), g_2(\cdot) \}$:

$\min \{ g_1(\lambda y + (1-\lambda) z), g_2(\lambda y + (1-\lambda) z) \} \geq \lambda \min \{ g_1(y), g_2(y) \} + (1-\lambda) \min \{ g_1(z), g_2(z) \}$

From the definition of $g_1(y)$ an $g_2(y)$, inequality (a) is concluded

then. Since by hypothesis: $y \succsim x$ and $z \succsim x$. Then:

$\lambda \min \{ y_1, 2y_2 \} \geq \lambda \min \{ x_1, 2x_2 \}$
$(1-\lambda) \min \{ z_1, 2z_2 \} \geq (1-\lambda) \min \{ x_1, 2x_2 \}$ (b)

$\Rightarrow \boxed{\lambda \cdot \min \{ y_1, 2y_2 \} + (1-\lambda) \min \{ z_1, 2z_2 \} \geq \min \{ x_1, 2x_2 \}}$

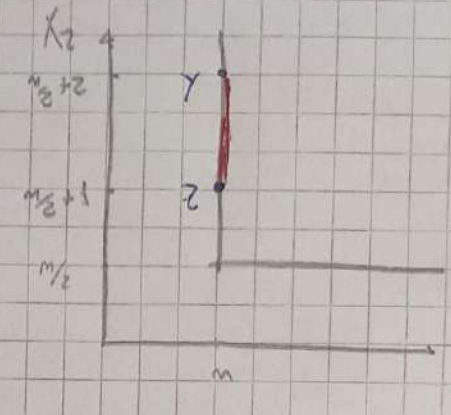
Inequalities (a) and (b) together imply by transitivity in $\mathbb{R}$+.

that $\lambda y + (1-\lambda) z \succsim x$, as desired

But not strictly convex: let $m = \min\{x_1, 2x_2\}$, where $x = (x_1, x_2)$

Consider: $y = (m, m/2+2)$

$z = (m, m/2+1)$



Bed segment: $\lambda y + (1-\lambda)z$, $\lambda \in [0,1]$

Since $\lambda y + (1-\lambda)z \in I(x)$,

we conclude that $\lambda y + (1-\lambda)z \not\succsim x$. Then $\succsim$ is not strictly convex

4. A regulation imposes $x_1 \le \overline{x_1}$. Explain how the restriction changes the set of undominated feasible bundles when electricity remains unrestricted.

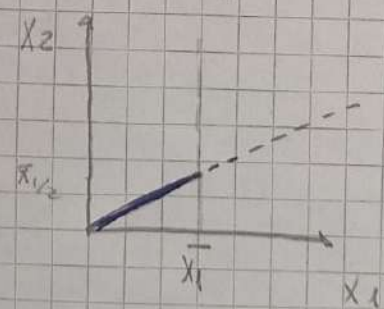
A. The set of undominated feasible bundles: $\{x : x_1 = 2x_2,\ x_1 \in \mathbb{R}\}$

the reason: for any $x_1' > x_1$: $(x_1', 2x_2) \in I(x)$, where $x = (x_1, x_2)$

If the restriction $x_1 \le \overline{x_1}$ is imposed, then:

the set of undominated feasible bundles is now: $\{x : x_1 = 2x_2,\ x_1 \le \overline{x_1}\}$

Graphically:



—: new set of undom feasible bundles after restriction

---: set of undom. feasible bundles before restriction