Production Theory

Technology, profit maximization, supply, and cost

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Lecture 6: technology and production sets

Objectives

- Represent production possibilities as subsets of commodity space.
- Relate production sets, production functions, and transformation functions.
- Study free disposal, convexity, returns to scale, and efficiency.

Reference

Mas-Colell, Whinston, and Green, Sections 5.A–5.B.

Production plans

A production plan is a vector $y = (y_1, \ldots, y_L) \in \mathbb{R}^L$.

- $y_\ell > 0$: the firm produces commodity ℓ.
- $y_\ell < 0$: the firm uses commodity ℓ as an input.
- $y_\ell = 0$: the plan neither produces nor uses commodity ℓ.

Example

With labor, steel, and cars ordered as commodities $1, 2, 3$,

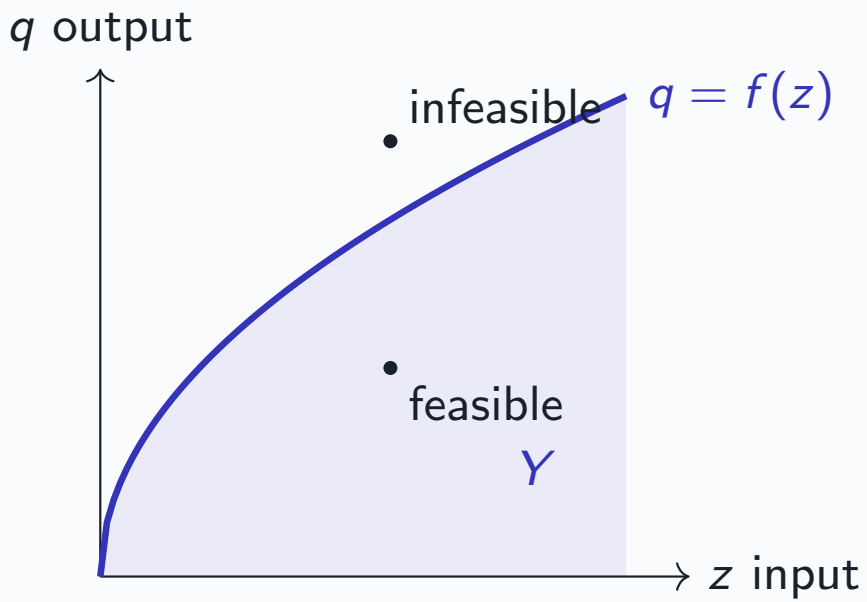
$$y = (-10, -4, 2)$$

uses 10 units of labor and 4 units of steel to produce 2 cars.

The sign convention lets inputs and outputs enter the same market-clearing equation.

The production set

The production set $Y \subset \mathbb{R}^L$ contains every technologically feasible plan.



Single input and output

Write a plan as $(q, -z)$. The shaded region is

$$Y = \{(q, -z) : z \geq 0,\ q \leq f(z)\}.$$

Points below the frontier waste some productive capacity but remain feasible under free disposal.

Production and transformation functions

Single output

For inputs $z \in \mathbb{R}_+^{L-1}$, the production function gives maximum output:

$$f(z) = \max\{q : (q, -z) \in Y\}, \qquad Y = \{(q, -z) : q \leq f(z)\}.$$

Several outputs

A transformation function $F : \mathbb{R}^L \to \mathbb{R}$ represents technology through

$$Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}, \qquad F(y) = 0 \text{ on the efficient frontier.}$$

The representation is not unique. If ϕ is increasing and $\phi(0) = 0$, then $\phi \circ F$ describes the same set.

Standard production functions

Technology	Production function	Main feature
Leontief	$f(z) = \min_i \{z_i / a_i\}$	Fixed input proportions
Linear	$f(z) = \sum_i a_i z_i$	Perfectly substitutable inputs
Cobb–Douglas	$f(z) = A \prod_i z_i^{\alpha_i}$	Smooth substitution; returns governed by $\sum_i \alpha_i$
CES	$f(z) = A\left(\sum_i a_i z_i^{\rho}\right)^{1/\rho}$	Constant elasticity $1/(1-\rho)$

For a single-output technology, convexity of Y is equivalent to concavity of f.

Regularity assumptions on production sets

Common assumptions on $Y \subset \mathbb{R}^L$ include:

Inaction $0 \in Y$.

No free production $Y \cap \mathbb{R}^L_+ \subset \{0\}$.

Irreversibility $Y \cap (-Y) \subset \{0\}$.

Free disposal $Y - \mathbb{R}^L_+ \subset Y$.

Convexity $\lambda y + (1-\lambda)y' \in Y$ for $y, y' \in Y$ and $\lambda \in [0,1]$.

Closedness Limits of feasible production plans remain feasible.

Convexity permits divisibility and replication. It rules out nonconvexities created by fixed costs or increasing returns.

Returns to scale

For a production set Y with $0 \in Y$:

Nonincreasing returns $y \in Y$ and $t \in [0, 1]$ imply $ty \in Y$.

Nondecreasing returns $y \in Y$ and $t \geq 1$ imply $ty \in Y$.

Constant returns $y \in Y$ and $t \geq 0$ imply $ty \in Y$.

For a production function, these conditions become

$$f(tz) \begin{cases} \leq tf(z), & t \geq 1 \quad \text{(nonincreasing returns)}, \\ \geq tf(z), & t \geq 1 \quad \text{(nondecreasing returns)}, \\ = tf(z), & t > 0 \quad \text{(constant returns)}. \end{cases}$$

Convexity and scale: two short proofs

Convexity plus inaction implies nonincreasing returns

For $y \in Y$ and $t \in [0, 1]$,

$$ty = ty + (1 - t)0 \in Y$$

because Y is convex and $0 \in Y$.

A convex constant-returns technology is a convex cone

Constant returns make Y a cone. For $y, y' \in Y$ and $a, b \geq 0$, if $a + b > 0$, convexity gives

$$\frac{a}{a+b}y + \frac{b}{a+b}y' \in Y.$$

Scaling by $a + b$ yields $ay + by' \in Y$.

Efficient production plans

Let $y \in Y$.

Efficient No $y' \in Y$ satisfies $y' \geq y$ and $y' \neq y$.

Weakly efficient No $y' \in Y$ satisfies $y' \gg y$.

Equivalently,

$$(y + \mathbb{R}^L_+) \cap Y = \{y\} \quad \text{and} \quad (y + \mathbb{R}^L_{++}) \cap Y = \varnothing.$$

Boundary characterization

If Y is closed and satisfies free disposal, every boundary point is weakly efficient and every weakly efficient point lies on the boundary.

Supporting prices and efficiency

Suppose Y is closed and convex. If $y^* \in \partial Y$, the supporting-hyperplane theorem gives a nonzero vector p such that

$$p \cdot y^* \geq p \cdot y \qquad \text{for every } y \in Y.$$

Under free disposal, every supporting price satisfies $p \geq 0$.

Economic interpretation

A weakly efficient plan maximizes the value of net output at some nonnegative price vector. If $p \gg 0$, any maximizer is efficient.

The result connects technological efficiency to profit maximization without assigning preferences to the firm.

Worked example: returns to scale in Cobb–Douglas production

Let

$$f(K, L) = AK^{\alpha} L^{\beta}, \qquad A > 0, \quad \alpha, \beta > 0.$$

Scaling both inputs by $t > 0$ gives

$$f(tK, tL) = t^{\alpha+\beta} f(K, L).$$

Therefore:

- $\alpha + \beta < 1$: decreasing returns to scale.
- $\alpha + \beta = 1$: constant returns to scale.
- $\alpha + \beta > 1$: increasing returns to scale.

The marginal products are positive, while concavity requires $\alpha + \beta \leq 1$.

Application exercise: technology

Consider

$$Y = \{(q, -z_1, -z_2) \in \mathbb{R}^3 : q \leq z_1^{1/3} z_2^{1/3},\ z_1, z_2 \geq 0\}.$$

1. Verify inaction, free disposal, closedness, and convexity.
2. Determine the returns to scale.
3. Characterize efficient plans with strictly positive output.
4. At an efficient plan, compute the marginal rate of technical substitution and interpret it geometrically.

Extension

Add a fixed cost $F > 0$ in units of good 2. Explain which properties of the production set fail and why competitive supply may become discontinuous.

Lectures 7–8: producer choice

Objectives

- Derive supply from profit maximization.
- Establish the main properties of supply and profit functions.
- Use cost minimization to obtain conditional factor demands.

Reference

Mas-Colell, Whinston, and Green, Sections 5.C–5.D.

The firm's problem

At prices $p \in \mathbb{R}^L_+ \setminus \{0\}$, the firm solves

$$\pi(p) = \sup_{y \in Y} p \cdot y.$$

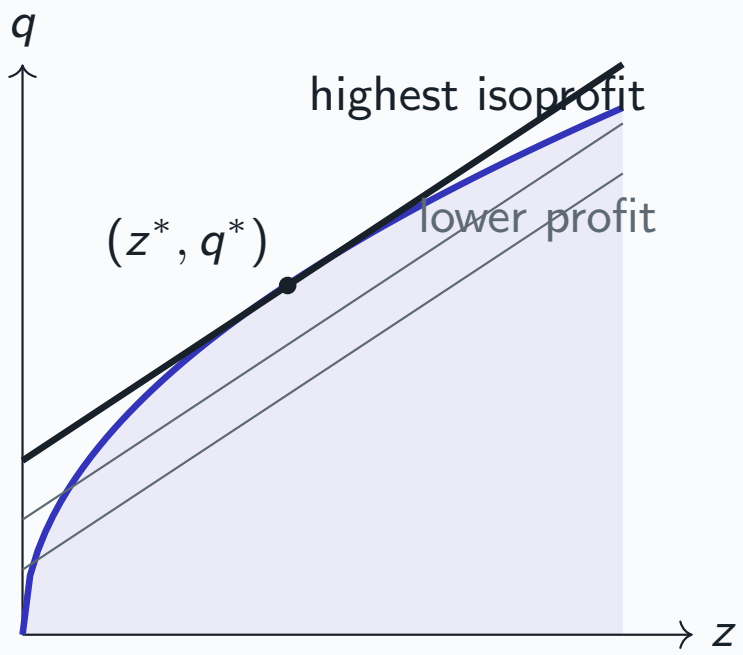
Its supply correspondence is

$$y(p) = \arg\max_{y \in Y} p \cdot y.$$

- Positive coordinates of $y(p)$ are output supplies.
- Negative coordinates are input demands.
- The maximizer may fail to exist, and profit may be unbounded.

Profit maximization is a model of the firm's market choice given its technology. It does not describe internal organization or ownership.

Profit maximization in the single-output case



With output price r and input price w,

$$\max_{z \geq 0} \; rf(z) - wz.$$

Isoprofit lines satisfy

$$q = \frac{\bar{\pi}}{r} + \frac{w}{r} z.$$

At an interior optimum,

$$rf'(z^*) = w.$$

Basic properties of supply

Assume the profit maximum exists.

1. If $p \gg 0$, every $y \in y(p)$ is efficient.
2. Supply is homogeneous of degree zero: $y(tp) = y(p)$ for $t > 0$.
3. If Y is convex, $y(p)$ is convex-valued.
4. If Y is strictly convex, supply is single-valued whenever it is nonempty.

Existence

Closedness alone does not ensure an optimizer. A sufficient condition is that the upper contour set $\{y \in Y : p \cdot y \geq a\}$ be compact for some feasible profit level a.

Proofs of the supply properties

Efficiency If $y' \geq y$, $y' \neq y$, and $p \gg 0$, then $p \cdot y' > p \cdot y$, contradicting optimality.

Homogeneity Multiplying prices by $t > 0$ multiplies every feasible profit by t and leaves the ranking of plans unchanged.

Convex-valuedness If $y, y' \in y(p)$ and Y is convex, then for $\lambda \in [0, 1]$,

$$p \cdot [\lambda y + (1 - \lambda)y'] = \lambda \pi(p) + (1 - \lambda)\pi(p) = \pi(p).$$

Uniqueness With strict convexity, two distinct maximizers would have a midpoint in the interior of Y. A small move in a positive-value direction would remain feasible and raise profit.

Properties of the profit function

The profit function is the support function of Y:

$$\pi(p) = \sup_{y \in Y} p \cdot y.$$

It is therefore:

- homogeneous of degree one: $\pi(tp) = t\pi(p)$ for $t > 0$;
- convex in prices;
- lower semicontinuous, and continuous wherever it is finite on an open set.

Economic content

Homogeneity reflects a change of monetary units. Convexity records the value of adapting the production plan when relative prices change.

Proof: homogeneity and convexity of profit

For $t > 0$,

$$\pi(tp) = \sup_{y \in Y} tp \cdot y = t \sup_{y \in Y} p \cdot y = t\pi(p).$$

Thus π is homogeneous of degree one. For $\lambda \in [0, 1]$,

$$\begin{aligned} \pi(\lambda p + (1-\lambda)p') &= \sup_{y \in Y} \{\lambda p \cdot y + (1-\lambda)p' \cdot y\} \\ &\leq \lambda \sup_{y \in Y} p \cdot y + (1-\lambda) \sup_{y \in Y} p' \cdot y \\ &= \lambda\pi(p) + (1-\lambda)\pi(p'). \end{aligned}$$

Hence π is convex. The inequality can be strict because the firm may use different production plans at p and p'.

Proof: lower semicontinuity of profit

For each fixed $y \in Y$, the function $p \mapsto p \cdot y$ is continuous and

$$\pi(p) = \sup_{y \in Y} p \cdot y.$$

For every $a \in \mathbb{R}$,

$$\{p : \pi(p) > a\} = \bigcup_{y \in Y} \{p : p \cdot y > a\}.$$

The sets on the right are open, so their union is open. Therefore π is lower semicontinuous.

Continuity on the finite domain

A finite convex function is continuous on the interior of its domain. Hence if π is finite on an open neighborhood of p, it is continuous at p.

Hotelling's lemma

Suppose π is finite near p and differentiable at p.

Lemma

Every profit-maximizing production plan equals the price gradient of profit:

$$\boxed{\nabla \pi(p) = y(p).}$$

Proof

For any $y \in y(p)$ and any p',

$$\pi(p') \geq p' \cdot y = \pi(p) + (p' - p) \cdot y.$$

Thus y is a subgradient of the convex function π at p. Differentiability makes the subgradient unique, so $y = \nabla \pi(p)$.

The law of supply

Let $y \in y(p)$ and $y' \in y(p')$. Optimality gives

$$p \cdot y \geq p \cdot y', \qquad p' \cdot y' \geq p' \cdot y.$$

Adding the inequalities yields

$$\boxed{(p' - p) \cdot (y' - y) \geq 0.}$$

If supply is differentiable, Hotelling's lemma implies

$$Dy(p) = D^2\pi(p),$$

so $Dy(p)$ is symmetric and positive semidefinite. Homogeneity also gives $Dy(p)p = 0$.

Constant returns to scale

Suppose Y is a cone and $0 \in Y$.

- If some $y \in Y$ yields $p \cdot y > 0$, then $p \cdot (ty) = tp \cdot y \to \infty$. Profit is unbounded.
- If $p \cdot y \leq 0$ for every $y \in Y$, then $\pi(p) = 0$ because the firm can choose 0.
- Any nonzero optimizer must earn zero profit, and every multiple of it is also optimal.

Competitive interpretation

At prices that support a constant-returns technology, equilibrium profit is zero and the firm's scale is indeterminate. Market clearing determines aggregate scale.

First-order conditions with a transformation function

Let $Y = \{y : F(y) \leq 0\}$, with F differentiable near a boundary point y^* and $\nabla F(y^*) \neq 0$. If y^* maximizes profit at $p \gg 0$, then

$$F(y^*) = 0, \qquad p = \lambda \nabla F(y^*) \quad \text{for some } \lambda > 0.$$

Hence for any two commodities i and j,

$$\frac{p_i}{p_j} = \frac{\partial F(y^*)/\partial y_i}{\partial F(y^*)/\partial y_j}.$$

If F is quasiconvex, these conditions are also sufficient for global profit maximization.

First-order conditions for input choice

With output price r, input prices $w = (w_1, \ldots, w_K)$, and production function f, the firm solves

$$\max_{z \geq 0} \; rf(z) - w \cdot z.$$

If f is differentiable and concave, the Kuhn–Tucker conditions are necessary and sufficient:

$$r\frac{\partial f(z^*)}{\partial z_i} \leq w_i, \qquad z_i^* > 0 \implies r\frac{\partial f(z^*)}{\partial z_i} = w_i.$$

For two inputs used in positive quantities,

$$\frac{f_i(z^*)}{f_j(z^*)} = \frac{w_i}{w_j}.$$

The marginal rate of technical substitution equals the input-price ratio.

Worked example: profit maximization

Let $f(z) = Az^{\alpha}$ with $A > 0$ and $0 < \alpha < 1$. The firm solves

$$\max_{z \geq 0} \; rAz^{\alpha} - wz.$$

The interior first-order condition gives

$$rA\alpha z^{\alpha-1} = w, \qquad \boxed{z^*(r,w) = \left(\frac{rA\alpha}{w}\right)^{1/(1-\alpha)}}.$$

Using $wz^* = rA\alpha(z^*)^{\alpha}$,

$$q^* = A(z^*)^{\alpha}, \qquad \pi(r,w) = (1-\alpha)rA(z^*)^{\alpha}.$$

Input demand rises with the output price and falls with the input price.

Cost minimization

For a target output q, the firm solves

$$c(w, q) = \min_{z \geq 0} \{w \cdot z : f(z) \geq q\}.$$

The solution $z(w, q)$ is the conditional factor-demand correspondence.

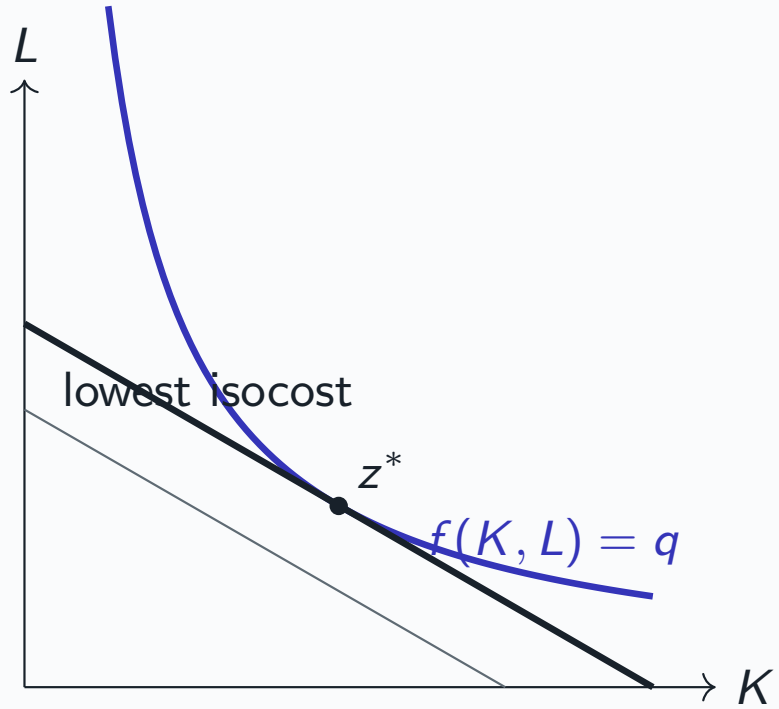
Why "conditional"?

Output is fixed at q. Profit maximization adds the choice of output:

$$\pi(r, w) = \sup_{q \geq 0} \{rq - c(w, q)\}.$$

Under monotonicity, a cost-minimizing bundle satisfies $f(z) = q$.

Cost minimization: isoquant and isocost



An isocost line satisfies

$$rK + wL = \bar{c}.$$

At an interior solution,

$$\frac{f_K(K^*, L^*)}{f_L(K^*, L^*)} = \frac{r}{w}.$$

Higher isocost lines lie farther from the origin.

Properties of the cost function

Under standard regularity conditions:

1. $c(w, q)$ is nondecreasing in q.
2. It is homogeneous of degree one in input prices: $c(tw, q) = tc(w, q)$ for $t > 0$.
3. It is concave in w because it is the pointwise infimum of functions linear in prices.
4. Conditional factor demand is homogeneous of degree zero in w.
5. If the technology is convex, $c(w, q)$ is convex in output.

Concavity in input prices implies that compensated own-price effects are nonpositive.

Proof: monotonicity and homogeneity of cost

Define the input requirement set

$$V(q) = \{z \geq 0 : f(z) \geq q\}, \qquad c(w, q) = \inf_{z \in V(q)} w \cdot z.$$

If $q' \geq q$, then $V(q') \subset V(q)$. Minimizing over the smaller set gives

$$c(w, q') \geq c(w, q).$$

For every $t > 0$,

$$c(tw, q) = \inf_{z \in V(q)} tw \cdot z = t \inf_{z \in V(q)} w \cdot z = tc(w, q).$$

Scaling all input prices preserves the ranking of feasible bundles, so

$$z(tw, q) = z(w, q).$$

Proof: concavity of cost in input prices

Let $w_\lambda = \lambda w + (1-\lambda)w'$ with $\lambda \in [0,1]$. Then

$$\begin{aligned} c(w_\lambda, q) &= \inf_{z \in V(q)} \{\lambda w \cdot z + (1-\lambda) w' \cdot z\} \\ &\geq \lambda \inf_{z \in V(q)} w \cdot z + (1-\lambda) \inf_{z \in V(q)} w' \cdot z \\ &= \lambda c(w, q) + (1-\lambda) c(w', q). \end{aligned}$$

Hence $c(\cdot, q)$ is concave.

Interpretation

When relative input prices change, the firm can substitute toward the cheaper input bundle. This adjustment keeps cost below the linear extrapolation from any fixed bundle.

Proof: convexity of cost in output

Suppose the technology is convex. Let $z \in V(q)$ and $z' \in V(q')$ be cost-minimizing bundles. For $\lambda \in [0,1]$, convexity of the technology implies

$$\lambda z + (1-\lambda)z' \in V(\lambda q + (1-\lambda)q').$$

Therefore

$$\begin{aligned} c(w, \lambda q + (1-\lambda)q') &\leq w \cdot [\lambda z + (1-\lambda)z'] \\ &= \lambda c(w,q) + (1-\lambda)c(w,q'). \end{aligned}$$

Thus $c(w,\cdot)$ is convex. The argument also works with minimizing sequences when a cost minimizer does not exist.

Shephard's lemma for production

Suppose conditional factor demand is single-valued and continuous at (w, q).

Lemma

$$\boxed{\nabla_w c(w, q) = z(w, q).}$$

First inequality

Let $z = z(w, q)$. Since this bundle remains feasible at any other price w',

$$c(w', q) \leq w' \cdot z = c(w, q) + (w' - w) \cdot z.$$

Thus $z(w, q)$ is a supergradient of the concave function $c(\cdot, q)$.

Proof of Shephard's lemma

Let $z = z(w, q)$ and $z' = z(w', q)$. Optimality at w and w' gives

$$c(w', q) - c(w, q) \leq (w' - w) \cdot z,$$

and

$$c(w', q) - c(w, q) \geq (w' - w) \cdot z'.$$

Set $w' = w + th$, with $t > 0$. Then

$$h \cdot z(w + th, q) \leq \frac{c(w + th, q) - c(w, q)}{t} \leq h \cdot z(w, q).$$

Continuity of conditional factor demand and $t \downarrow 0$ imply

$$D_w c(w, q)h = h \cdot z(w, q) \quad \text{for every direction } h.$$

Hence c is differentiable in input prices and $\boxed{\nabla_w c(w, q) = z(w, q)}$.

Comparative statics of conditional factor demand

If c is twice differentiable, Shephard's lemma gives

$$D_w z(w,q) = D^2_{ww} c(w,q).$$

Since c is concave in input prices,

$$D_w z = (D_w z)^\top, \qquad a^\top D_w z\, a \leq 0 \quad \text{for every } a.$$

Therefore:

- $\partial z_i / \partial w_i \leq 0$ for every input i;
- compensated cross-price effects are symmetric;
- homogeneity implies $D_w z(w,q) w = 0$.

Worked example: Cobb–Douglas cost minimization

Let $f(K, L) = AK^{\alpha} L^{1-\alpha}$ with $0 < \alpha < 1$. Capital and labor prices are r and w. The first-order condition gives

$$\frac{L}{K} = \frac{1-\alpha}{\alpha} \frac{r}{w}.$$

Substitution into the output constraint yields the cost function

$$\boxed{c(r, w, q) = \frac{q}{A} \left(\frac{r}{\alpha}\right)^{\alpha} \left(\frac{w}{1-\alpha}\right)^{1-\alpha}.}$$

Shephard's lemma gives conditional demands:

$$K(r, w, q) = \frac{\alpha c(r, w, q)}{r}, \qquad L(r, w, q) = \frac{(1-\alpha) c(r, w, q)}{w}.$$

Output choice through the cost function

After solving the conditional problem, the firm chooses output from

$$\max_{q \geq 0} \; rq - c(w, q).$$

At an interior optimum,

$$r = \frac{\partial c(w, q^*)}{\partial q}.$$

Constant returns

If $c(w, q) = q\, c(w, 1)$, then

- $r < c(w, 1)$ implies $q^* = 0$;
- $r = c(w, 1)$ makes every scale optimal;
- $r > c(w, 1)$ makes profit unbounded.

This reproduces the zero-profit result for constant-returns technologies.

Envelope logic: Hotelling and Shephard

Both lemmas follow from the same principle.

Problem	Value function	Price derivative
Profit maximization	$\pi(p) = \max_{y \in Y} p \cdot y$	$\nabla \pi(p) = y(p)$
Cost minimization	$c(w, q) = \min_{z: f(z) \geq q} w \cdot z$	$\nabla_w c(w, q) = z(w, q)$

The derivative of the optimized value with respect to a price equals the corresponding optimized net supply or conditional input demand.

Application exercise: producer choice

A firm has technology

$$f(K, L) = K^{1/4} L^{1/2}.$$

Output sells at price p, while capital and labor cost r and w.

1. Determine returns to scale and verify concavity.
2. Derive conditional factor demands and the cost function.
3. Solve for profit-maximizing output and unconditional factor demands.
4. Verify Hotelling's lemma and Shephard's lemma directly.
5. Show that the Jacobian of conditional factor demand is symmetric and negative semidefinite.

Key results

- Production sets represent feasible net-output vectors and make multiple-input, multiple-output analysis possible.
- Convexity supports averaging and decentralization by prices. Constant returns turn the technology into a cone.
- Profit is a convex support function. Hotelling's lemma recovers supply from its price gradient.
- Cost is concave in input prices. Shephard's lemma recovers conditional factor demand from its price gradient.
- Convex duality generates symmetry and semidefiniteness restrictions that observed firm behavior must satisfy.