Competitive Equilibrium

Exchange, production, welfare, and the core

Francis Bloch - Antoine Mandel
francis.bloch@univ-paris1.fr

2026–2027

Lecture 9: competitive equilibrium in exchange

- How can decentralized choices be mutually compatible?
- How do prices coordinate consumers who begin with different endowments?
- When does a competitive equilibrium exist?

Road map

Feasibility $\rightarrow$ individual demand $\rightarrow$ market clearing $\rightarrow$ excess demand $\rightarrow$ existence.

References: MWG, Sections 15.A–15.B and 17.C.

Exchange economy: primitives and notation

There are I consumers and L commodities.

- Consumer i has consumption set $X_i = \mathbb{R}^L_+$, preference $\succsim_i$, and endowment $\omega_i \in \mathbb{R}^L_+$.
- Aggregate resources are $\bar{\omega} = \sum_{i=1}^{I} \omega_i \gg 0$.
- At prices $p \in \mathbb{R}^L_+ \setminus \{0\}$, wealth is $w_i(p) = p \cdot \omega_i$ and

$$B_i(p) = \{x_i \in \mathbb{R}^L_+ : p \cdot x_i \leq p \cdot \omega_i\}.$$

- If preferences admit utility u_i, Walrasian demand is

$$x_i(p) \in \arg \max_{x_i \in B_i(p)} u_i(x_i).$$

Only relative prices matter: $B_i(tp) = B_i(p)$ for every $t > 0$.

Feasible allocations

Definition

An allocation $x = (x_1, \ldots, x_I)$ is *feasible* if

$$x_i \in \mathbb{R}^L_+ \quad \text{and} \quad \sum_{i=1}^{I} x_i \leq \bar{\omega}.$$

The feasible set is denoted $A(\bar{\omega})$.

Proposition

$A(\bar{\omega})$ is nonempty, compact, and convex.

Economic meaning

Feasibility imposes only the aggregate resource constraint. It does not require individual optimality, voluntary participation, or supporting prices.

Feasible allocations: proof of the properties

Proof.

$0 \in A(\bar{\omega})$, so the set is nonempty. It is closed because it is defined by finitely many weak inequalities. For every i, h, feasibility implies

$$0 \leq x_{ih} \leq \bar{\omega}_h.$$

Hence $A(\bar{\omega})$ is bounded and therefore compact. If $x, y \in A(\bar{\omega})$ and $t \in [0, 1]$, then

$$\sum_i [tx_i + (1-t)y_i] \leq t\bar{\omega} + (1-t)\bar{\omega} = \bar{\omega},$$

so $tx + (1-t)y$ is feasible and the set is convex. □

Walrasian equilibrium

Definition

A *Walrasian equilibrium* is a price vector $p^* \neq 0$ and an allocation x^* such that

1. for every i, $x_i^* \in x_i(p^*)$;
2. every market clears:

$$\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} \omega_i = \bar{\omega}.$$

- Consumers take prices as given; they do not coordinate directly.
- Price normalization is harmless: set $p_L = 1$ or $\sum_h p_h = 1$.
- With free disposal, local nonsatiation and positive prices eliminate waste.

Budget exhaustion and individual rationality

Proposition

If preferences are locally nonsatiated, every demanded bundle exhausts wealth:

$$p \cdot x_i(p) = p \cdot \omega_i.$$

Consequently, an equilibrium allocation is individually rational: $x_i^* \succsim_i \omega_i$.

Proof.

Suppose $p \cdot x_i < p \cdot \omega_i$. The strict slack leaves an open neighborhood of x_i inside the budget set. Local nonsatiation gives a preferred bundle x_i' in that neighborhood, contradicting optimality. Thus equality holds. Since the endowment itself is affordable, $\omega_i \in B_i(p^*)$; optimality then gives $x_i^* \succsim_i \omega_i$. □

Why equilibrium prices are strictly positive

Why strictly positive equilibrium prices?

If preferences are strictly monotone and good h has positive aggregate supply, $p_h^* = 0$ would make additional units of good h free. Demand could not have a finite optimum, so $p_h^* > 0$.

Proof.

Take any candidate optimum x_i with finite coordinates. If $p_h = 0$, then for every $a > 0$ the bundle $x_i + ae_h$ has exactly the same expenditure. Strict monotonicity implies $x_i + ae_h \succ_i x_i$, so x_i cannot maximize utility. Thus a finite equilibrium demand requires $p_h > 0$. □

Excess demand and Walras' law

Aggregate excess demand is

$$z(p) = \sum_{i=1}^{I} x_i(p) - \bar{\omega}.$$

Under locally nonsatiated preferences,

$$\begin{aligned} p \cdot z(p) &= \sum_i p \cdot x_i(p) - p \cdot \sum_i \omega_i \\ &= \sum_i p \cdot \omega_i - \sum_i p \cdot \omega_i = 0. \end{aligned}$$

Walras' law

The value of aggregate excess demand is zero at every price vector, not only at equilibrium.

If $p \gg 0$ and $L - 1$ markets clear, then

$$0 = p \cdot z(p) = p_L z_L(p),$$

so the last market clears automatically.

Properties of aggregate excess demand

Assume each demand is single-valued and continuous.

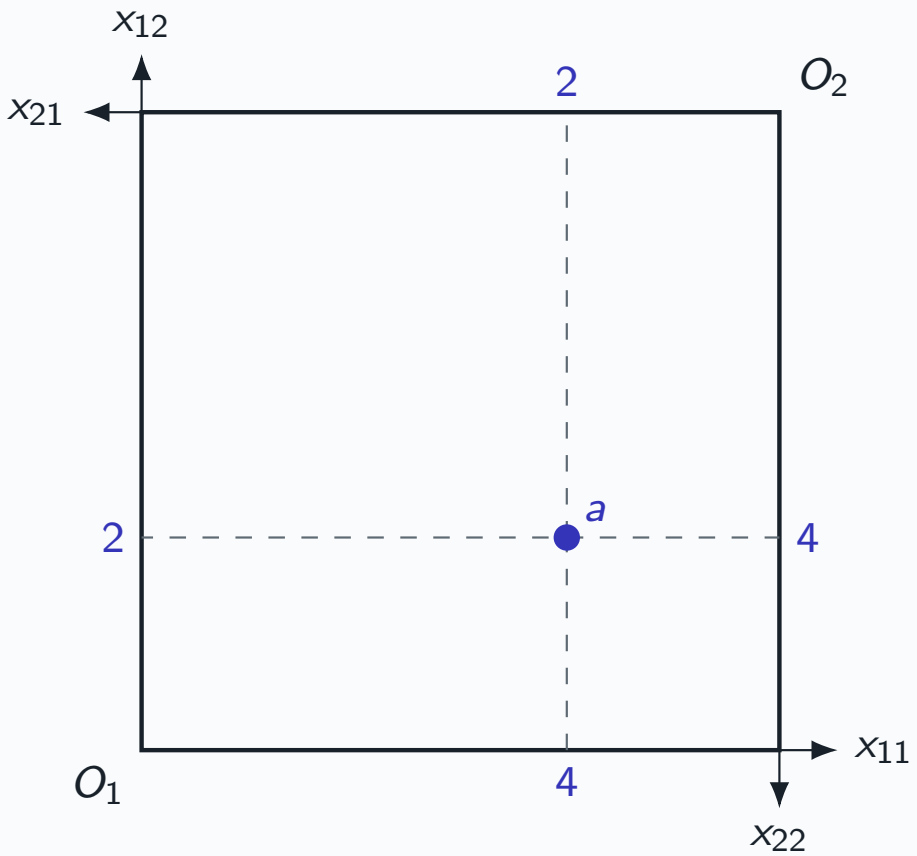
1. **Homogeneity of degree zero:** $z(tp) = z(p)$ for $t > 0$, because multiplying all prices and wealth by t leaves every budget set unchanged.
2. **Continuity:** each $x_i(p)$ is continuous by the maximum theorem; finite sums preserve continuity.
3. **Walras' law:** $p \cdot z(p) = 0$, by budget exhaustion.
4. **Boundary behavior:** if preferences are strongly monotone and $p_h^n \downarrow 0$ while wealth stays positive, desired consumption of good h becomes unbounded; hence $\|z(p^n)\| \to \infty$.

These properties reduce equilibrium to finding a zero of a continuous vector field on the price simplex.

Edgeworth box: one feasible allocation

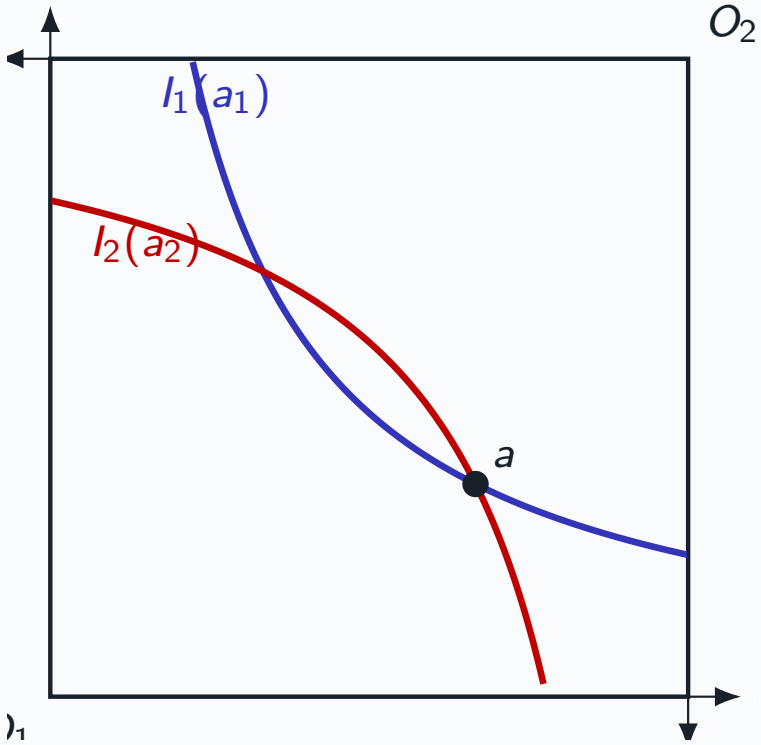
Let $\bar{\omega} = (6, 6)$ and $u_i(x_{i1}, x_{i2}) = x_{i1}x_{i2}$. The point $a = (4, 2)$ denotes the complete allocation

$$a_1 = (4, 2), \qquad a_2 = \bar{\omega} - a_1 = (2, 4).$$



Indifference curves through the allocation

At $a = (4, 2)$, both utilities equal 8. Consumer 1's indifference curve is $x_{11}x_{12} = 8$; consumer 2's curve is $(6 - x_{11})(6 - x_{12}) = 8$.

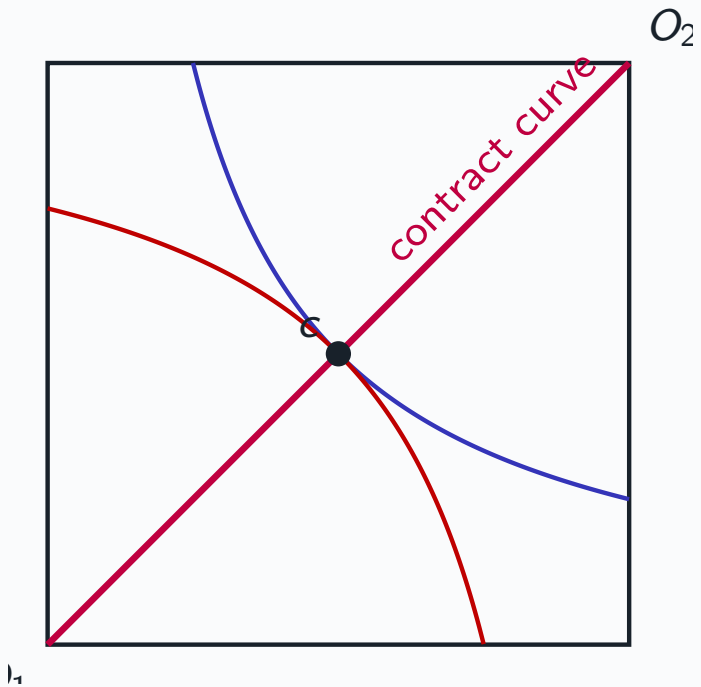


The contract curve

For the symmetric Cobb–Douglas economy,

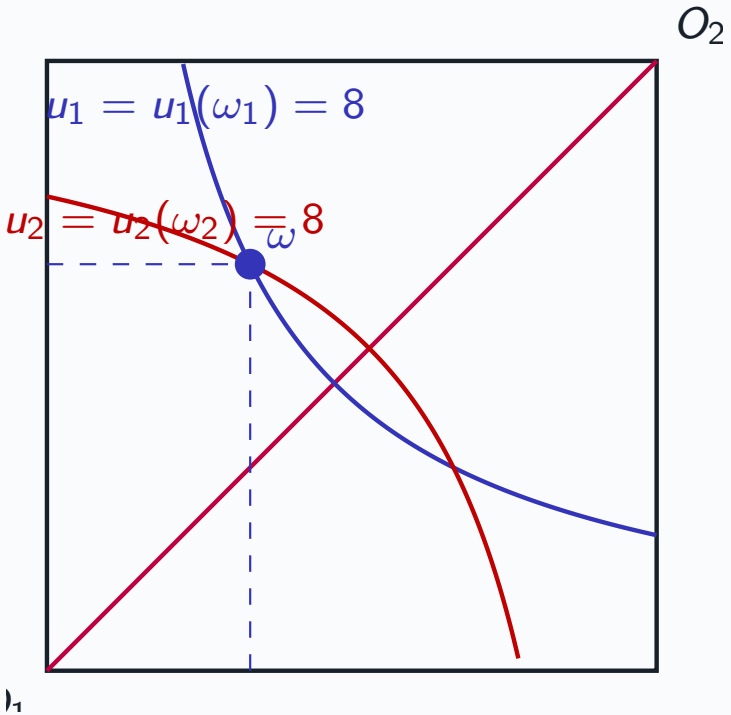
$$MRS_1 = \frac{x_{12}}{x_{11}}, \qquad MRS_2 = \frac{6 - x_{12}}{6 - x_{11}}.$$

Equality of MRSs gives $x_{12} = x_{11}$. The diagonal is therefore the set of interior Pareto-optimal allocations.



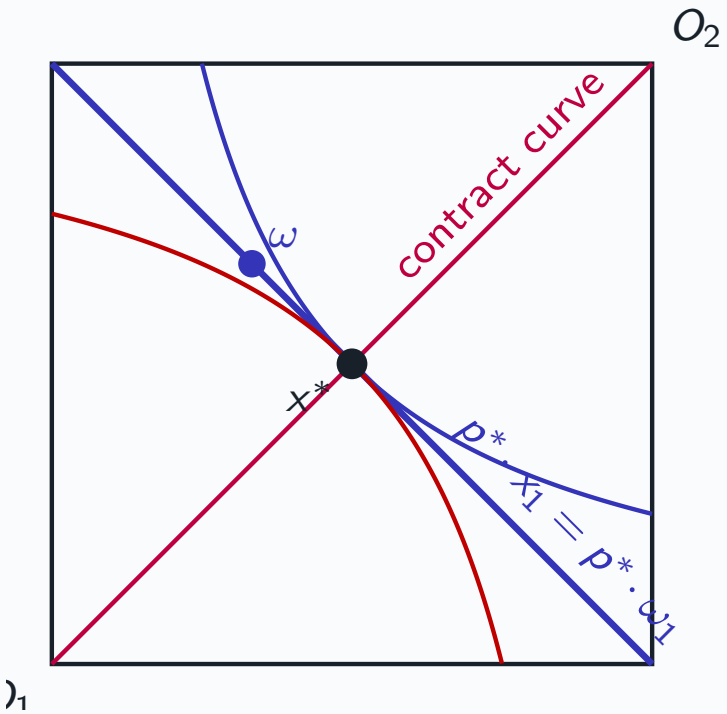
The initial endowment

Let consumer 1's endowment be $\omega_1 = (2, 4)$, so consumer 2 owns $\omega_2 = (4, 2)$. The endowment point is $\omega = (2, 4)$ in consumer 1's coordinates.



The competitive allocation

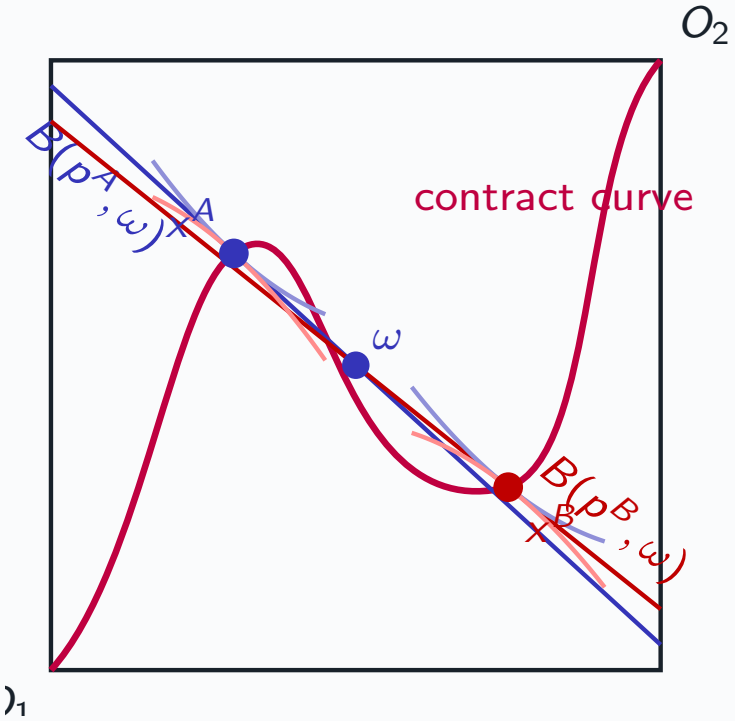
At $p^* = (1, 1)$, consumer 1 has wealth 6 and demands $(3, 3)$. Consumer 2 does the same. The budget line through ω is $x_{11} + x_{12} = 6$.



$$MRS_1(x_1^*) = 1 = \frac{p_1^*}{p_2^*} = MRS_2(x_2^*).$$

Two competitive equilibria can coexist

The diagram represents a regular economy with a nonmonotone contract curve. Each budget line passes through the same endowment and supports both consumers at its marked allocation.



Multiplicity

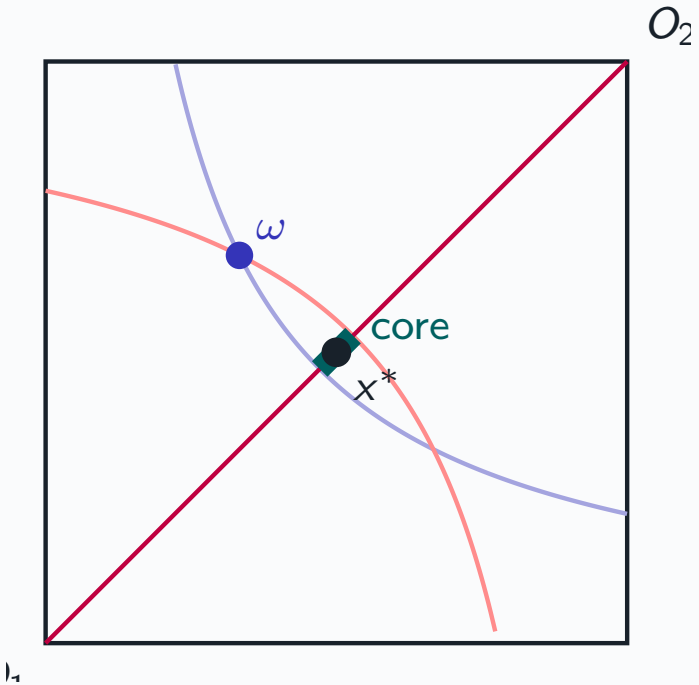
Both x^A and x^B are Pareto optimal and satisfy market clearing, but they are supported by different relative prices. Existence does not imply uniqueness.

The core in the Edgeworth box

At $\omega = (2, 4)$, an allocation (t, t) on the contract curve is individually rational when

$$t^2 \geq 8 \quad \text{and} \quad (6 - t)^2 \geq 8.$$

Hence the core is the segment $t \in [\sqrt{8}, 6 - \sqrt{8}]$.



Application: sharing aggregate risk

Interpret the two goods as consumption in two states, $s = 1, 2$.

- Consumer i has state-contingent endowment $(\omega_{i1}, \omega_{i2})$ and expected utility
$$U_i(c_i) = \pi u_i(c_{i1}) + (1 - \pi) u_i(c_{i2}).$$
- An Arrow security for state s delivers one unit only if state s occurs. Its price is p_s.
- A competitive equilibrium trades state-contingent claims before uncertainty is resolved.

Interior optimality gives

$$\frac{\pi u_i'(c_{i1})}{(1 - \pi) u_i'(c_{i2})} = \frac{p_1}{p_2} \qquad \text{for every } i.$$

Hence agents' probability-weighted marginal utilities are aligned. The Edgeworth box is a picture of equilibrium insurance: idiosyncratic risk is pooled, while aggregate risk remains.

Worked exercise: a Cobb–Douglas exchange economy

Two consumers have identical utility

$$u_i(x_{i1}, x_{i2}) = x_{i1}^{\alpha} x_{i2}^{1-\alpha}, \qquad 0 < \alpha < 1,$$

and endowments $\omega_1 = (1, 2)$, $\omega_2 = (2, 1)$.

Questions

Normalize $p_2 = 1$. Find the equilibrium relative price and allocation. Verify market clearing.

The Marshallian demands are

$$x_{i1} = \alpha \frac{m_i}{p_1}, \qquad x_{i2} = (1 - \alpha) m_i,$$

where $m_1 = p_1 + 2$ and $m_2 = 2p_1 + 1$.

Worked exercise: solving for the price

Aggregate demand for good 1 is

$$\begin{aligned} x_{11} + x_{21} &= \frac{\alpha}{p_1}(m_1 + m_2) \\ &= \frac{\alpha}{p_1}(3p_1 + 3). \end{aligned}$$

Equating demand to aggregate supply 3 gives

$$\alpha \frac{3p_1 + 3}{p_1} = 3 \quad \Longrightarrow \quad p_1^* = \frac{\alpha}{1 - \alpha}.$$

Walras' law implies that good 2 clears once good 1 clears, but we will verify it directly.

Worked exercise: allocation and verification

At $p_1^* = \alpha/(1-\alpha)$,

$$x_1^* = (2-\alpha, 2-\alpha), \qquad x_2^* = (1+\alpha, 1+\alpha).$$

Indeed,

$$x_1^* + x_2^* = (3,3) = \omega_1 + \omega_2.$$

Each budget also balances:

$$p^* \cdot x_1^* = p^* \cdot \omega_1, \qquad p^* \cdot x_2^* = p^* \cdot \omega_2.$$

For $\alpha = 1/2$, $p^* = (1,1)$ and both consumers receive $(3/2, 3/2)$: trade completely insures the asymmetric endowments.

Existence in two goods

Normalize $p_2 = 1$ and let $z_1(p_1)$ denote excess demand for good 1.

1. z_1 is continuous when demand is continuous.
2. As $p_1 \downarrow 0$, the boundary condition gives $z_1(p_1) > 0$.
3. For a sufficiently large p_1, bounded wealth and finite resources give $z_1(p_1) < 0$.
4. The intermediate value theorem yields $p_1^* > 0$ with $z_1(p_1^*) = 0$.
5. Walras' law then gives
$$p_1^* z_1(p^*) + z_2(p^*) = 0,$$
so $z_2(p^*) = 0$.

The general proof replaces the intermediate value theorem with a fixed-point theorem.

Existence theorem for an exchange economy

Theorem

Let $X_i = \mathbb{R}^L_+$. Suppose each $\succsim_i$ is complete, transitive, continuous, convex, and strongly monotone, and each endowment satisfies $\omega_i \gg 0$. Then a Walrasian equilibrium exists.

For the explicit proof, first add strict convexity, so each Walrasian demand $x_i(p)$ is single-valued. The proof has four parts:

1. establish the properties of aggregate excess demand;
2. keep prices away from zero on a truncated simplex;
3. apply Brouwer to a projected price-adjustment map;
4. let the truncation vanish and prove market clearing.

The last slide explains the extension to convex, non-strict preferences.

Step 1: properties of excess demand

On $\Delta_{++} = \{p \gg 0 : \sum_h p_h = 1\}$, let

$$z(p) = \sum_i x_i(p) - \bar{\omega}.$$

1. Budgets are nonempty and compact. Continuity and strict convexity give a unique maximizer. Berge's maximum theorem makes x_i, hence z, continuous.
2. Homogeneity of degree zero follows because $B_i(tp) = B_i(p)$.
3. Strong monotonicity implies budget exhaustion. Therefore

$$p \cdot z(p) = \sum_i [p \cdot x_i(p) - p \cdot \omega_i] = 0.$$

4. Since consumption is nonnegative,

$$z_h(p) \geq -\bar{\omega}_h.$$

Thus excess demand is bounded below component by component.

Step 2: behavior near the boundary

Lemma (Boundary condition)

If $p^n \in \Delta_{++}$ converges to $\bar{p} \in \partial\Delta$, then

$$\max_h z_h(p^n) \longrightarrow +\infty.$$

Proof.

Some price tends to zero, while another remains positive because prices sum to one. Since $\omega_i \gg 0$, limiting wealth is positive. If a demand sequence were bounded, it would have a limit $\bar{x}_i$ in the limiting budget set. Adding a small amount of a zero-price good gives a strictly preferred bundle. For large n, an arbitrarily small reduction in a positive-price good makes this bundle affordable at p^n, while continuity preserves the strict preference. This contradicts optimality. Positive-price goods are uniformly bounded by the budget, so demand must diverge for a good whose price approaches zero. □

This is the step that prevents the fixed-point sequence from converging to a boundary price.

Step 3: a fixed point on a truncated simplex

For $n > L$, define the compact convex set

$$\Delta_n = \left\{ p \in \Delta : p_h \geq \frac{1}{n} \text{ for every } h \right\}.$$

Let Π_{Δ_n} be Euclidean projection onto Δ_n and define

$$T_n(p) = \Pi_{\Delta_n}[p + z(p)].$$

T_n is continuous and maps Δ_n into itself. Brouwer's theorem gives $p^n = T_n(p^n)$. The projection characterization says

$$(q - p^n) \cdot z(p^n) \leq 0 \qquad \text{for every } q \in \Delta_n.$$

By Walras' law, $p^n \cdot z(p^n) = 0$, so

$$q \cdot z(p^n) \leq 0 \qquad \text{for every } q \in \Delta_n. \tag{$*$}$$

Step 4: fixed points stay away from the boundary

Suppose a subsequence p^n converged to $\partial\Delta$. By the boundary condition and finiteness of L, a further subsequence has one fixed component $z_h(p^n)$ tending to $+\infty$. Choose $q^{n,h} \in \Delta_n$ by

$$q_h^{n,h} = 1 - \frac{L-1}{n}, \qquad q_k^{n,h} = \frac{1}{n} \quad (k \neq h).$$

The negative components of z are bounded below by $-\bar{\omega}_k$. Hence

$$\begin{aligned} q^{n,h} \cdot z(p^n) &= \left(1 - \frac{L-1}{n}\right) z_h(p^n) + \frac{1}{n} \sum_{k \neq h} z_k(p^n) \\ &\geq \left(1 - \frac{L-1}{n}\right) z_h(p^n) - \frac{1}{n} \sum_{k \neq h} \bar{\omega}_k \longrightarrow +\infty. \end{aligned}$$

This contradicts inequality $(*)$. Therefore a subsequence converges to some $p^* \gg 0$.

Step 5: the limit clears every market

For each good h, use $q^{n,h}$ in inequality $(*)$:

$$q^{n,h} \cdot z(p^n) \leq 0.$$

Because $p^n \to p^* \gg 0$, continuity makes $z(p^n) \to z(p^*)$, while $q^{n,h} \to e_h$. Taking limits gives

$$z_h(p^*) \leq 0 \qquad \text{for every } h.$$

Walras' law and strict positivity of prices now imply

$$0 = p^* \cdot z(p^*) = \sum_h p_h^* z_h(p^*) \quad \Longrightarrow \quad z_h(p^*) = 0 \quad \forall h.$$

Thus p^* clears every market, and $x_i(p^*)$ is optimal in every consumer's budget set. This proves existence under strict convexity.

Convex preferences: the Kakutani extension

Without strict convexity, demand $D_i(p)$ may contain several bundles. Continuity and convex preferences imply that D_i is nonempty, upper hemicontinuous, and convex-valued on every Δ_n.
Replace the single-valued map with the correspondence

$$\Phi_n(p, x) = \Pi_{\Delta_n}\left[p + \sum_i x_i - \bar{\omega}\right] \times \prod_i D_i(p),$$

on a sufficiently large compact product of prices and consumption bundles. Kakutani's theorem gives a fixed point (p^n, x^n).
The same projection inequality, boundary argument, and limiting argument yield $p^* \gg 0$ and

$$\sum_i x_i^* = \bar{\omega}, \qquad x_i^* \in D_i(p^*) \quad \forall i.$$

Hence convexity, rather than strict convexity, is enough for existence.

Lecture 9: what to retain

- An equilibrium combines individual optimality with market clearing.
- Walras' law follows from budget exhaustion and makes one market equation redundant.
- The Edgeworth box shows feasibility, mutually beneficial trade, and equilibrium tangency in one picture.
- Existence is a fixed-point result; continuity, convexity, monotonicity, and boundary behavior do distinct jobs.
- Competitive trade can be interpreted as intertemporal exchange or trade in state-contingent claims.

Lecture 10: private-ownership production economies

- Consumers supply inputs, buy outputs, and receive profit income.
- Firms choose production plans to maximize profits at market prices.
- Prices must make consumer and firm plans jointly feasible.

Road map

Definition → equilibrium properties → marginal conditions → existence and nonconvexities.

References: MWG, Sections 16.A–16.C and 17.C.

The private-ownership economy

There are I consumers, J firms, and L commodities.

- Consumer i has $(X_i, \succsim_i, \omega_i)$ and owns share $\theta_{ij} \geq 0$ of firm j, with $\sum_i \theta_{ij} = 1$.
- Firm j has production set $Y_j \subset \mathbb{R}^L$. A plan y_j records net outputs: positive components are outputs and negative components are inputs.
- Firm j's profit function and supply correspondence are

$$\pi_j(p) = \max_{y_j \in Y_j} p \cdot y_j, \qquad y_j(p) \in \arg\max_{y_j \in Y_j} p \cdot y_j.$$

- Consumer i's wealth is

$$m_i(p) = p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} \pi_j(p).$$

Equilibrium with production

Definition

A private-ownership competitive equilibrium (p^*, x^*, y^*) satisfies:

1. **Profit maximization:**
$$y_j^* \in \arg\max_{y_j \in Y_j} p^* \cdot y_j \quad \text{for every } j.$$
2. **Utility maximization:**
$$x_i^* \in \arg\max_{x_i \in X_i} \left\{ u_i(x_i) : p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_j \theta_{ij} p^* \cdot y_j^* \right\}.$$
3. **Market clearing:**
$$\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*.$$

Basic properties: proof

Proposition

Under local nonsatiation, an equilibrium has the following properties.

1. *Prices can be normalized:* (p^*, x^*, y^*) *and* (tp^*, x^*, y^*) *are equivalent for* $t > 0$.
2. *Every consumer exhausts wealth.*
3. *Every consumer is at least as well off as at the endowment if inactivity* $0 \in Y_j$ *for every firm.*

Basic properties: proof

Proof.

Scaling p scales all budget inequalities and profits by the same t, leaving argmax sets unchanged. Budget exhaustion follows from the same slack-plus-local-nonsatiation argument as in exchange. If $0 \in Y_j$, profit maximization gives $\pi_j(p^*) \geq 0$; hence $p^* \cdot \omega_i \leq m_i(p^*)$. The endowment is affordable, so optimality yields $x_i^* \succsim_i \omega_i$. □

Walras' law with production

Define aggregate excess demand by

$$z(p) = \sum_i x_i(p) - \sum_i \omega_i - \sum_j y_j(p).$$

Budget exhaustion and full distribution of profits imply

$$\begin{aligned} p \cdot \sum_i x_i(p) &= p \cdot \sum_i \omega_i + \sum_i \sum_j \theta_{ij} \pi_j(p) \\ &= p \cdot \sum_i \omega_i + \sum_j \pi_j(p) \\ &= p \cdot \sum_i \omega_i + p \cdot \sum_j y_j(p). \end{aligned}$$

Therefore $p \cdot z(p) = 0$. If $p \gg 0$ and all but one market clear, the remaining market clears as well.

Marginal conditions at an interior equilibrium

For consumer i, the Lagrangian first-order conditions give

$$\nabla u_i(x_i^*) = \lambda_i p^*.$$

Thus every consumer has the same marginal rate of substitution:

$$\frac{\partial u_i/\partial x_h}{\partial u_i/\partial x_k} = \frac{p_h^*}{p_k^*}.$$

For a differentiable transformation frontier $F_j(y_j) = 0$, profit maximization gives

$$p^* = \mu_j \nabla F_j(y_j^*),$$

so the firm's marginal rate of transformation also equals the price ratio. Hence

$$MRS_i^{hk} = \frac{p_h^*}{p_k^*} = MRT_j^{hk}.$$

Prices align marginal willingness to substitute with marginal technological trade-offs.

Application: labor, output, and dividend income

Take consumption as good 1 and labor as a negative component of good 2.

- Households choose consumption and leisure; their labor endowment finances consumption.
- Firms hire labor and sell output; households receive profits according to ownership shares.
- The wage is simultaneously a household opportunity cost of leisure and a firm input price.

In equilibrium,

$$\underbrace{\sum_i (\text{time endowment} - \ell_i)}_{\text{labor supplied}} = \underbrace{\sum_j n_j}_{\text{labor demanded}}, \qquad \sum_i c_i = \sum_j q_j.$$

This model links the product market, labor market, and distribution of firm profits without a separate coordinating institution.

Worked exercise: one household and one firm

The household has one unit of time, utility

$$u(c, \ell) = \ln c + \ln \ell,$$

and owns the firm. The firm produces $q = \sqrt{n}$ using labor n. Output is the numeraire and the wage is w.

Questions

Find the firm's choice, household choice, equilibrium wage, and allocation.

The firm's problem is

$$\max_{n \geq 0} \{\sqrt{n} - wn\}.$$

The first-order condition $1/(2\sqrt{n}) = w$ gives

$$n(w) = \frac{1}{4w^2}, \qquad \pi(w) = \frac{1}{4w}.$$

Worked exercise: household choice and clearing

The household budget is

$$c + w\ell \leq w + \pi(w).$$

With Cobb–Douglas utility, half of income $M = w + \pi(w)$ is spent on each good:

$$c(w) = \frac{M}{2}, \qquad \ell(w) = \frac{M}{2w} = \frac{1}{2} + \frac{1}{8w^2}.$$

Labor supply is $1 - \ell(w)$. Labor-market clearing requires

$$\frac{1}{4w^2} = \frac{1}{2} - \frac{1}{8w^2} \quad \Longrightarrow \quad w^* = \frac{\sqrt{3}}{2}.$$

Worked exercise: equilibrium allocation

At $w^* = \sqrt{3}/2$,

$$n^* = \frac{1}{3}, \qquad \ell^* = \frac{2}{3}, \qquad q^* = \frac{1}{\sqrt{3}},$$

and

$$c^* = \frac{w^* + \pi(w^*)}{2} = \frac{1}{\sqrt{3}} = q^*.$$

Thus both labor and output markets clear. Notice the accounting role of profits: labor income alone is not household income because the household also owns the firm.

Marginal condition

At the solution, the household's marginal willingness to exchange leisure for consumption equals the wage, and the firm's marginal product of labor also equals the wage.

Existence with production

Theorem (Informal)

An equilibrium exists when consumer preferences satisfy the standard continuity, convexity, and nonsatiation assumptions; production sets are closed and convex, allow inactivity, rule out free production, and aggregate feasible allocations are bounded.

Roles of the assumptions

Convexity keeps choices convex-valued; closedness and bounded feasible allocations provide compactness; no free production prevents resources from being created without inputs.

Existence with production: proof architecture

Proof architecture

1. Normalize prices to a compact simplex and truncate consumption and production to a sufficiently large compact set.
2. The maximum theorem gives upper-hemicontinuous, nonempty, convex-valued consumer demand and firm supply correspondences.
3. Combine optimal choices with a price correspondence that raises prices of goods in excess demand.
4. Kakutani yields a fixed point; Walras' law and boundary arguments imply market clearing.
5. Boundedness shows the truncation never binds, so the fixed point is an equilibrium of the original economy.

Why convexity matters

With a fixed cost, a firm's production set may be nonconvex. For example,

$$Y = \{(0,0)\} \cup \{(y,-n) : y \leq a(n-\bar{n}),\ n \geq \bar{n}\},$$

where $\bar{n} > 0$ is a fixed labor requirement.

- Supply can jump when price crosses average cost.
- A market-clearing quantity may lie between "shut down" and the minimum scale.
- Marginal-cost pricing may not cover fixed cost.

Thus competitive equilibrium may fail to exist, and supporting an efficient allocation may require nonlinear prices, subsidies, or regulation. Natural monopoly is the canonical application.

Lecture 10: what to retain

- Profit income enters consumer wealth; ownership closes the circular flow.
- Walras' law survives production because all profits are distributed.
- At an interior equilibrium, consumers' MRSs and firms' MRTs equal common price ratios.
- Existence extends the exchange argument but requires restrictions preventing unbounded production and free output.
- Nonconvex technologies are economically important and can invalidate price-taking decentralization.

Lecture 11: Pareto optimality and welfare theorems

- Which feasible allocations are efficient?
- Why is a competitive equilibrium Pareto optimal?
- Which Pareto optima can be decentralized by prices and transfers?

Road map

Planner problems $\rightarrow$ marginal efficiency $\rightarrow$ First Welfare Theorem $\rightarrow$ Second Welfare Theorem.

References: MWG, Sections 10.B, 16.D, and 16.E.

Pareto optimality

Definition

A feasible allocation x^* is *Pareto optimal* if there is no feasible x such that

$$x_i \succsim_i x_i^* \quad \text{for every } i, \qquad x_k \succ_k x_k^* \quad \text{for some } k.$$

In an exchange economy, feasibility is $\sum_i x_i \le \bar{\omega}$. With production, it is

$$\sum_i x_i \le \sum_i \omega_i + \sum_j y_j, \qquad y_j \in Y_j.$$

Pareto optimality is silent about distribution: it rules out unanimous improvements but does not rank efficient allocations.

Existence of a Pareto optimum

Proposition

Suppose the feasible set A is nonempty and compact and utilities are continuous. For any weights $\lambda_i > 0$, every solution of

$$\max_{x \in A} \sum_{i=1}^{I} \lambda_i u_i(x_i)$$

is Pareto optimal.

Existence of a Pareto optimum: proof

Proof.

Continuity of the weighted sum and compactness of A give a maximizer x^* by Weierstrass' theorem. If feasible x Pareto dominated x^*, then $u_i(x_i) \geq u_i(x_i^*)$ for all i and the inequality would be strict for at least one consumer. Since every $\lambda_i > 0$,

$$\sum_i \lambda_i u_i(x_i) > \sum_i \lambda_i u_i(x_i^*),$$

contradicting optimality. □

Constrained-utility characterization

Fix promised utility levels $\bar{u}_2, \ldots, \bar{u}_I$. Consider

$$\max_{x \in A} u_1(x_1) \quad \text{s.t.} \quad u_i(x_i) \geq \bar{u}_i, \quad i = 2, \ldots, I.$$

Proposition

A feasible allocation x^* is Pareto optimal if and only if it solves this problem when $\bar{u}_i = u_i(x_i^*)$ for $i \geq 2$.

This characterization does not attach interpersonal meaning to utility levels: it holds the others at their reference indifference curves and pushes consumer 1 as high as feasibility permits.

Constrained-utility characterization: proof

Necessity. If x^* is Pareto optimal but does not solve the problem, there is feasible x with

$$u_1(x_1) > u_1(x_1^*), \qquad u_i(x_i) \geq u_i(x_i^*) \quad (i \geq 2),$$

which Pareto dominates x^*, a contradiction.

Sufficiency. Suppose x^* solves the problem and some feasible x Pareto dominates it. If consumer 1 is strictly better, x directly contradicts maximality. If another consumer is strictly better while consumer 1 is only indifferent, local nonsatiation and continuity permit a small feasible reallocation that preserves the others' promised utilities and makes consumer 1 strictly better. This again contradicts maximality.

Marginal efficiency in exchange

For two consumers and two goods, maximize $u_1(x_1)$ subject to

$$u_2(\bar{\omega} - x_1) \geq \bar{u}_2.$$

At an interior solution, the Lagrangian

$$\mathcal{L} = u_1(x_1) + \mu[u_2(\bar{\omega} - x_1) - \bar{u}_2]$$

gives, for $h = 1, 2$,

$$\frac{\partial u_1}{\partial x_{1h}} = \mu \frac{\partial u_2}{\partial x_{2h}}.$$

Dividing the conditions for the two goods yields

$$MRS_1^{12} = MRS_2^{12}.$$

If MRSs differed, a small trade could make both consumers better off; equality is the no-gains-from-trade condition.

Marginal efficiency with production

For simplicity, one firm transforms input 2 into output 1. The planner chooses (x, y) subject to

$$\sum_i x_i \leq \bar{\omega} + y, \qquad y \in Y.$$

The multipliers q_h on resource constraints act as shadow prices. Interior first-order conditions imply

$$\nabla u_i(x_i^*) = \lambda_i q, \qquad q \text{ supports } Y \text{ at } y^*.$$

Therefore

$$MRS_i^{12} = \frac{q_1}{q_2} = MRT^{12} \quad \text{for every } i.$$

Efficiency requires both exchange efficiency across consumers and production efficiency between inputs and outputs.

The feasible utility set and Negishi weights

Define the attainable utility set

$$\mathcal{U} = \{v \in \mathbb{R}^I : \exists x \in A,\ v_i \le u_i(x_i)\ \forall i\}.$$

If utilities are concave and the feasible set is convex, $\mathcal{U}$ is convex after the usual comprehensive closure.

Theorem (Negishi characterization)

Every maximizer of $\sum_i \lambda_i u_i(x_i)$ with $\lambda_i > 0$ is Pareto optimal. Conversely, every supported Pareto optimum maximizes such a weighted sum for some $\lambda \ge 0$.

Proof sketch. Sufficiency is the positive-weight contradiction proved earlier. For necessity, a supporting-hyperplane theorem separates the convex set $\mathcal{U}$ from utility vectors strictly above the Pareto point. The normal vector of the supporting hyperplane supplies the welfare weights λ.

First Welfare Theorem

Theorem

If preferences are locally nonsatiated, every competitive equilibrium allocation in an exchange economy is Pareto optimal.

Proof, step 1. Let (p^*, x^*) be an equilibrium. Suppose feasible x Pareto dominates x^*. Revealed preference at the optimum implies

$$x_i \succ_i x_i^* \quad \Longrightarrow \quad p^* \cdot x_i > p^* \cdot x_i^*.$$

For a consumer only weakly better off, local nonsatiation implies

$$p^* \cdot x_i \geq p^* \cdot x_i^*;$$

otherwise a nearby strictly preferred bundle would remain affordable, contradicting optimality.

First Welfare Theorem: contradiction

Proof, step 2. Summing the expenditure inequalities, with at least one strict inequality, gives

$$p^* \cdot \sum_i x_i > p^* \cdot \sum_i x_i^*.$$

But feasibility of x and nonnegative prices imply

$$p^* \cdot \sum_i x_i \leq p^* \cdot \bar{\omega} = p^* \cdot \sum_i x_i^*,$$

a contradiction. Hence no feasible allocation Pareto dominates x^*.

Interpretation

The theorem is an efficiency result, not an equity result. It also does not claim existence or uniqueness of equilibrium.

Second Welfare Theorem: precise statement

Consider a pure exchange economy with $X_i = \mathbb{R}^L_+$. Suppose every preference relation is continuous, convex, and locally nonsatiated. Let x^* be a Pareto-optimal allocation satisfying

$$\sum_i x_i^* = \bar{\omega} \quad \text{and} \quad x_i^* \gg 0 \quad \text{for every } i.$$

Theorem (Second Welfare Theorem)

There exist a nonzero price vector $p^* \geq 0$ and balanced lump-sum transfers $(T_i)_i$ such that

$$x_i^* \in \arg\max_{x_i \in X_i} \{x_i : \ p^* \cdot x_i \leq p^* \cdot \omega_i + T_i\}, \qquad \sum_i T_i = 0.$$

Thus (p^*, x^*) is a competitive equilibrium for the redistributed economy.

The interiority assumption gives every consumer strictly positive wealth. Without it, the separation argument below always yields a *price quasi-equilibrium*; an additional qualification is needed to obtain full utility maximization.

Second Welfare Theorem: Step 1, preferred net trades

For each consumer define the strict upper contour set at the target allocation:

$$P_i(x_i^*) = \{x_i \in X_i : x_i \succ_i x_i^*\}.$$

Continuity makes $P_i(x_i^*)$ relatively open. Convex preferences make it convex. Consider the set of aggregate resources required by simultaneous strict improvements:

$$C = \sum_{i=1}^{I} \left(P_i(x_i^*) - x_i^*\right) + \mathbb{R}^L_{++}.$$

The last term is the set of strictly positive disposal vectors. The set C is therefore convex and open. Moreover,

$$0 \notin C.$$

Indeed, if $0 = \sum_i (y_i - x_i^*) + r$ for $y_i \succ_i x_i^*$ and $r \gg 0$, then

$$\sum_i y_i = \sum_i x_i^* - r \leq \bar{\omega}.$$

The feasible allocation $(y_i)_i$ would make every consumer strictly better off, contradicting Pareto optimality of x^*.

Second Welfare Theorem: Step 2, separation

Because C is nonempty, open, and convex and $0 \notin C$, the separating hyperplane theorem gives $p^* \neq 0$ and a scalar α such that

$$p^* \cdot c > \alpha \geq p^* \cdot 0 = 0 \qquad \text{for every } c \in C.$$

Since $C + \mathbb{R}^L_+ = C$, the separating vector must satisfy $p^* \geq 0$. To see this, if $p^*_h < 0$ for some good h, then $c + te_h \in C$ for every $t > 0$, while

$$p^* \cdot (c + te_h) = p^* \cdot c + tp^*_h \longrightarrow -\infty,$$

which contradicts separation.
Taking a sequence in C converging to a boundary point shows the weak support inequality

$$p^* \cdot c \geq 0 \qquad \text{for every } c \in \overline{C}.$$

Prices therefore measure the resource cost of every possible collection of strict improvements.

Second Welfare Theorem: Step 3, one price supports every consumer

Fix consumer i and a bundle $y_i \succ_i x_i^*$. Local nonsatiation lets us choose, for every $j \neq i$, a sequence

$$y_j^n \succ_j x_j^*, \qquad y_j^n \longrightarrow x_j^*.$$

For each n,

$$c^n = (y_i - x_i^*) + \sum_{j \neq i} (y_j^n - x_j^*) \in \overline{C}.$$

For every $r \gg 0$, $c^n + r \in C$; hence $c^n \in \overline{C}$. The weak support inequality gives $p^* \cdot c^n \geq 0$. Passing to the limit yields

$$y_i \succ_i x_i^* \quad \Longrightarrow \quad p^* \cdot y_i \geq p^* \cdot x_i^*. \tag{$\star$}$$

The same vector p^* therefore supports every consumer's upper contour set. In particular, no consumer can obtain a strictly preferred bundle at a cost strictly below the value of the target bundle. This is the consumer-optimality condition of a price quasi-equilibrium.

Second Welfare Theorem: Step 4, strict expenditure inequality

Set $m_i^* = p^* \cdot x_i^*$. Since $p^* \geq 0$, $p^* \neq 0$, and $x_i^* \gg 0$, we have $m_i^* > 0$.
We now strengthen $(\star)$ to

$$y_i \succ_i x_i^* \quad \Longrightarrow \quad p^* \cdot y_i > m_i^* . \tag{$\star\star$}$$

Suppose instead that $y_i \succ_i x_i^*$ and $p^* \cdot y_i = m_i^*$. For $\lambda \in (0,1)$ define $y_i^\lambda = \lambda y_i$. Continuity of preferences and $y_i \succ_i x_i^*$ imply that, for $\lambda < 1$ sufficiently close to one,

$$y_i^\lambda \succ_i x_i^* .$$

But its expenditure is

$$p^* \cdot y_i^\lambda = \lambda m_i^* < m_i^* ,$$

contradicting $(\star)$. Hence $(\star\star)$ holds. Every bundle in the budget set $\{x_i : p^* \cdot x_i \leq m_i^*\}$ is therefore weakly worse than x_i^*, so x_i^* solves consumer i's utility-maximization problem.

Second Welfare Theorem: Step 5, transfers and market clearing

Give consumer i the lump-sum transfer

$$T_i = p^* \cdot x_i^* - p^* \cdot \omega_i .$$

The resulting wealth equals the value of the target bundle:

$$p^* \cdot \omega_i + T_i = p^* \cdot x_i^* = m_i^* .$$

Step 4 therefore proves that x_i^* is optimal in consumer i's new budget set. The transfers balance because the target allocation is feasible with equality:

$$\begin{aligned} \sum_i T_i &= p^* \cdot \sum_i x_i^* - p^* \cdot \sum_i \omega_i \\ &= p^* \cdot \bar{\omega} - p^* \cdot \bar{\omega} = 0. \end{aligned}$$

Finally, $\sum_i x_i^* = \sum_i \omega_i$, so every market clears. We have shown consumer optimality, balanced redistribution, and market clearing. Hence (p^*, x^*) is a competitive equilibrium after lump-sum transfers. □

Second Welfare Theorem: what each assumption does

- **Convex preferences** make the improvement set convex, so a linear price hyperplane can support it.
- **Continuity** makes strict upper contour sets open and permits the limit and scaling arguments.
- **Local nonsatiation** supplies nearby strictly preferred bundles for the consumers held at their target bundles.
- **Interiority of** x_i^* ensures $p^* \cdot x_i^* > 0$, converting quasi-optimality into ordinary budget-set optimality.
- **Lump-sum transfers** change wealth without changing marginal trade-offs. They select the desired point on the Pareto frontier.

Without convexity, an efficient allocation may have no supporting price. Without the wealth qualification, a zero-wealth consumer may satisfy the quasi-equilibrium inequalities but fail to maximize utility on the budget set.

Application: separating equity from efficiency

The welfare theorems suggest a two-stage policy logic:

1. use lump-sum transfers to choose a point on the Pareto frontier;
2. let competitive prices decentralize that point.

This is conceptually powerful but practically limited.

- True lump-sum transfers cannot depend on behavior or hidden characteristics.
- Distortionary taxes change marginal incentives and may move the economy inside the frontier.
- Nonconvexities, externalities, public goods, market power, and incomplete markets break the hypotheses.

The theorem is therefore a benchmark for diagnosing which missing market or policy constraint creates an inefficiency.

Worked exercise: a Pareto frontier

Two consumers share total endowment $(1, 1)$ and have

$$u_i(x_i) = \ln x_{i1} + \ln x_{i2}.$$

For welfare weight $\lambda \in (0, 1)$, solve

$$\max_x \ \lambda u_1(x_1) + (1-\lambda)u_2(x_2) \quad \text{s.t.} \quad x_{1h} + x_{2h} = 1.$$

For each good h, the first-order condition is

$$\frac{\lambda}{x_{1h}} = \frac{1-\lambda}{1-x_{1h}},$$

so

$$x_1^* = (\lambda, \lambda), \qquad x_2^* = (1-\lambda, 1-\lambda).$$

Worked exercise: decentralization by transfers

Let $p = (1, 1)$ and initial endowments be $\omega_1 = (1, 0)$, $\omega_2 = (0, 1)$. With equal Cobb–Douglas exponents, a consumer with wealth m_i demands $(m_i/2, m_i/2)$. To decentralize the target allocation, assign wealth

$$m_1 = 2\lambda, \qquad m_2 = 2(1 - \lambda).$$

The required lump-sum transfers are

$$T_1 = 2\lambda - 1, \qquad T_2 = 1 - 2\lambda, \qquad T_1 + T_2 = 0.$$

For $\lambda = 3/4$, transfer $1/2$ unit of value to consumer 1. Demands become

$$x_1 = (3/4, 3/4), \qquad x_2 = (1/4, 1/4),$$

which clear both markets and reproduce the planner's allocation.

First Welfare Theorem with production

Theorem

Under local nonsatiation, every private-ownership competitive equilibrium allocation is Pareto optimal.

Proof. Suppose feasible (x, y) Pareto dominates equilibrium (x^*, y^*). Consumer optimality implies

$$p^* \cdot \sum_i x_i > p^* \cdot \sum_i x_i^* .$$

Firm optimality gives $p^* \cdot y_j \leq p^* \cdot y_j^*$ for every j. Feasibility then implies

$$\begin{aligned} p^* \cdot \sum_i x_i &\leq p^* \cdot \left(\sum_i \omega_i + \sum_j y_j \right) \\ &\leq p^* \cdot \left(\sum_i \omega_i + \sum_j y_j^* \right) = p^* \cdot \sum_i x_i^* , \end{aligned}$$

a contradiction.

Second Welfare Theorem with production

Theorem (Informal)

With convex preferences and convex production sets, every regular Pareto optimum can be supported by prices and decentralized through transfers and ownership shares.

Proof sketch.

1. Separate the convex set of technologically feasible net trades from the set of trades that would make consumers better off.
2. The separating vector p supports each firm's production set, so y_j^* maximizes profit.
3. The same p supports each consumer's upper contour set, so x_i^* is optimal at wealth $p \cdot x_i^*$.
4. Choose transfers and profit shares so each consumer receives exactly this wealth and aggregate transfers sum to zero.

Convexity is essential: without a supporting price hyperplane, an efficient production plan need not be profit maximizing at any linear price.

Coalitions and the core

In a pure exchange economy, a coalition $S \subseteq \{1, \ldots, I\}$ can use only the endowments of its members.

Definition

Coalition S *blocks* a feasible allocation x if it can find bundles $(y_i)_{i \in S}$ such that

$$\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i,$$

with $y_i \succsim_i x_i$ for every $i \in S$ and $y_k \succ_k x_k$ for at least one $k \in S$.

Definition

The *core* is the set of feasible allocations that no coalition blocks.

The grand coalition rules out Pareto-inefficient allocations. Smaller coalitions impose additional participation constraints. In the two-consumer Edgeworth box, the core is the individually rational part of the contract curve.

Every competitive allocation belongs to the core

Theorem

If preferences are locally nonsatiated, a Walrasian equilibrium allocation x^* belongs to the core.

Proof.

Suppose coalition S blocks x^* with $(y_i)_{i \in S}$. Consumer optimality and local nonsatiation imply

$$p^* \cdot y_i \geq p^* \cdot x_i^* = p^* \cdot \omega_i \quad (i \in S),$$

with a strict inequality for at least one member. Summing over S gives

$$p^* \cdot \sum_{i \in S} y_i > p^* \cdot \sum_{i \in S} \omega_i.$$

But the coalition's resource constraint and $p^* \geq 0$ imply the reverse weak inequality. This contradiction proves that no coalition blocks x^*. □

From the core to competitive equilibrium

The reverse inclusion need not hold in a finite economy: the core can contain many noncompetitive allocations. Replication makes coalitional deviations more powerful.

Theorem (Debreu–Scarf core convergence)

Replicate every consumer type r times. Under continuity, convexity, and strong monotonicity, the core allocations of the replicated economies converge, as $r \to \infty$, to the set of Walrasian allocations.

- A large coalition can reproduce almost any profitable market trade.
- A noncompetitive core allocation eventually becomes blockable.
- In an atomless economy, the core and the set of competitive allocations coincide under standard assumptions.

Thus price-taking behavior can be interpreted as the limiting discipline imposed by unrestricted coalition formation in a large economy.

Lecture 11: what to retain

- Pareto optimality means that no feasible unanimous improvement remains.
- Weighted welfare maximization traces supported points of the Pareto frontier.
- The First Welfare Theorem derives efficiency from consumer and firm optimality plus market clearing.
- The Second Welfare Theorem uses convexity and separation to decentralize efficient allocations after redistribution.
- Every competitive allocation lies in the core; in large replicated economies, the core converges to the competitive set.
- The theorems clarify both the power and the limits of competitive markets: efficiency does not choose a distribution, and failures of the assumptions matter.