NICHOLAS DENYER

11 Sun and Line
The Role of the Good

"I understand that less than I understand the Good of Plato," says a slave in comedy.[1] The slave's understanding of the Good would not have been much helped by attending Plato's own public lecture on the subject: the majority of the audience "came in the expectation of acquiring some of those things that conventionally count as human goods, such as health, wealth, strength, and in general some wonderful happiness; but when the discussion turned out to be on the mathematical sciences – numbers, geometry, astronomy – and the conclusion to be that good is one, this struck them as utterly paradoxical; whereupon some came to despise the business, and others started to make complaints."[2] Nevertheless, in one respect at least, it is actually quite easy to grasp what Plato has to say about the Good.

If for the moment we confine our attention to Forms of artifacts, it is easy to understand and accept the *Republic*'s claim that the Good has the privileged position of being what accounts for the existence and intelligibility of Forms, much as the Sun has the privileged position of being what accounts for the growth and visibility of plants (508b–e, 509b). For the claim will then be that everything about an ideal artifact is teleologically explicable. The ideal wheel is circular, and it has its axle in its center. These things are so because that is the best way for a wheel to be: a buckled wheel, or for that matter a

[1] Amphis, *Amphicrates* fr. 6, in Kassel 1983–.

[2] Aristoxenus, *Harmonics* 39.8–40.4 Rios, relaying a favorite anecdote of Aristotle's. An alternative translation of the words here translated as "and the conclusion to be that good is one" would be "and that the Limit is good, a single thing." For some discussion of these and other alternatives, see David Sedley's chapter 10 in this volume, note 21.

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circular wheel with its axle off-center, would give a bumpy ride. And the Good accounts in this way for every aspect of the ideal wheel: if, for example, we ask about the color of the ideal wheel, then the only answer is that it has no particular color; and that is because there is no one color that is the best one for a wheel to have. Moreover, in appreciating these facts about the ideal wheel we come to understand why it is as it is. And that is how the Good accounts not only for the ideal wheel's existence but also for its intelligibility.

We can appreciate quite how thorough is the teleological explanation of the ideal wheel if we compare it with the teleological explanation of ordinary wheels. First, some ordinary wheels are actually defective, and such defects (missing spokes, flat tires) would not be defects if they could be explained as contributing to the good that wheels are there to achieve. Second, every ordinary wheel has many features that, even though they are not defects, are nevertheless not explicable teleologically: for example, some ordinary wheels are lubricated by an odd number of molecules of lubricant, and the rest are lubricated by an even number; there is nothing bad about either of these; nor however is there anything good, by reference to which we might explain why the number of molecules of lubricant lubricating an ordinary wheel is odd rather than even, or even rather than odd. Third, even when there is a teleological explanation of some aspect of an ordinary wheel, that explanation will not be the full explanation: we can say, for example, that the wheel is lubricated in order for it to spin freely, as any good wheel must; but even here there will always be more to say, about why this wheel is properly lubricated, when others are not.

When, in the light of the Good, we come to understand a Form, we must, claims the analogy with the Sun, use our minds, not our senses (507b). This claim too is easy to understand and accept if we confine our attention to Forms of artifacts. Think of the history of bad design. Until quite recently, electric kettles had broad bases; they were made of metal; and they had their handles at the top. In an electric kettle, these features are vices: the broad base means that even if you wish to boil only a cupful of water, you must put in much more than that, to cover the electric element safely; metal conducts heat so well that the outside rapidly reaches the dangerous temperature of the water boiling within; and handles at the top catch more heat than handles at the side would do. These features, vices

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in an electric kettle, are virtues in a kettle that is to be boiled on a fire. The kettle's broad base means that it will catch more of the available heat; being made of metal, it will rapidly conduct that heat to the water within; and the handle at the top will catch less heat than one at the side. This accounts for the design of kettles in the era when the most convenient way to boil a kettle was by placing it on a fire. When, however, the era of electricity dawned, then instead of focusing their minds on what a kettle is, on the Form of a kettle, the designers of the first electric kettles simply copied what they could perceive of earlier kettles, and added an electric element.

In thinking about ideal artifacts, and understanding them in the light of the Good that makes them be as they are, we are not thinking directly about the Good, any more than we look at the sun when, by its light, we see the plants that its light causes to grow. Indeed, much as we would bedazzle our eyes in attempting to look directly at the Sun, we are likely to befuddle our minds in attempting to think directly about the Good. The ease with which we appreciated that the ideal wheel is circular, with its axle at the center, has no counterpart when we ask what the Form of Good itself is like. Nevertheless, we can say some things about the Form of Good. As the audience at Plato's lecture on the subject learned to their disappointment, the Form of Good is not going to be any of the things that conventionally count as human goods, such as health or wealth or strength. Nor, for that matter, is the Form of Good going to be knowledge or pleasure (505b–d). For what the ideal wheel and the ideal kettle have in common is obviously not health, wealth, strength, knowledge, or pleasure, but something far more abstract. And anything that makes ordinary ideal objects so clearly comprehensible, while remaining so obscure itself, cannot itself be any ordinary ideal object. It must be altogether more grand (509b).

These points about our teleological understanding of ideal artifacts are too obvious to be seriously contested. It is therefore important to see how they might fall short of more contestable aspects of the theory of Forms.

First, the sort of eternity that, in the light of teleological considerations, we can attach to ideal artifacts is not perhaps the full eternity that Plato would attach to Forms. In no case can we set a date to the beginning or end of an ideal artifact: we cannot say that the ideal car was built in 1999 – and that is not because it was built in some

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other year. In all cases, the truths about ideal artifacts that we can discern by the light of the Good are everlastingly and unshakably true: it is, for example, everlastingly and unshakably true that the ideal kettle boils water, whereas no such thing holds everlastingly and unshakably of any perceptible kettle. Moreover, in many cases, the truths that hold changelessly about ideal artifacts are themselves truths about immunity to change: the ideal kettle does not corrode; the ideal knife never goes blunt. But this is not so in all cases. Sometimes the unchangeable truth about an ideal artifact requires that the artifact change: since the ideal clock always gives the right time, the time that it gives must always be changing. Sometimes, indeed, the unchangeable truth about an ideal artifact will speak of how the artifact perishes: in the light of teleological considerations about what it takes to make a good bomb, we can and must say that the ideal bomb explodes when, and only when, it is detonated. However, this fact about the ideal bomb is itself a timeless fact; and neither this, nor any other fact about the ideal bomb that we can discern in the light of the Good, enables us to name the date when the ideal bomb exploded, or even to attach any sense to the claim that, for example, the ideal bomb will explode at noon on New Year's Day 2009. And it may be that this is all that is needed for the ideal bomb to have the sort of eternity that Plato would wish to ascribe to his Forms. After all, *Republic* 597c–d speaks of another ideal artifact, the ideal bed, as made by the ideal artificer, God. But some have felt this passage to be an aberration, out of keeping with the general tendency of the theory of Forms; indeed, Aristotle tells us of Platonists, very close to Plato himself, who denied outright that there are Forms of artifacts (*Metaphysics* 991b6–7, 1080a5–6), and it would be hard to prove that they were wholly unfaithful to Plato.

There is a second respect in which our teleological understanding of ideal artifacts may fall short of the full theory of Forms. The theory seems to claim more than just that, for example, the ideal sword is perfectly sharp and free from corrosion, and therefore not to be identified with any of the many perceptible swords, such as Arthur's sword Excalibur and Roland's sword Durendal. For the theory seems to make the further claim that such a thing is true because there exists, in addition to Excalibur and Durendal, another sword that, unlike them, is precisely as we know the ideal sword to be. And that this is a further claim is clear when we reflect on the age of

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the average undergraduate. The age of any perceptible undergraduate increases at the rate of one year for every twelve months. No perceptible undergraduate could therefore be identical with the average undergraduate, whose age has remained roughly stable over the past century, and whose age actually decreases when young students are admitted at the start of the academic year, and again when the oldest students graduate. But none of this requires us to posit an imperceptible undergraduate who, unlike perceptible undergraduates, does not age at the standard rate. The reason is obvious: "The average undergraduate is 19.8 years old" does not say of some extraordinary undergraduate what "Clara is 19.8 years old" says of an ordinary one; instead, it speaks of ordinary undergraduates, but in an extraordinary way. It may be that something similar is true of our talk about ideal artifacts. Or again, it may not. For to reformulate "The ideal sword is perfectly sharp" as a statement about Excalibur and Durendal and the like, rather than as saying truly about an imperceptible sword what "Excalibur is perfectly sharp" says falsely about Excalibur, would be, to put it mildly, harder than reformulating talk about the age of the average undergraduate as talk about the ages of the many undergraduates that we can perceive. And perhaps one of the claims of the full theory of Forms is that ideal artifacts are not in any such way reducible to their perceptible counterparts. Or again, perhaps not. For contrast the ideal sword with, for example, the ideal artifact whose name begins with an S. The trouble with this is not simply that there is no further artifact whose name begins with an S, over and above the various perceptible artifacts with such a name, like perceptible staplers and swords and saucepans; for the ideal artifact whose name begins with an S suffers from the graver trouble that it was a nonsense right from the start, and that there never was any such thing in the first place to be reduced to perceptible ones. The ideal sword, by contrast, whether or not it is a further sword, in addition to perceptible swords, is free from the graver trouble; and it may be that this is all that the theory of Forms requires.

Third, ideal artifacts are not the only ideal objects, distinct from perceptible objects that also bear their names. There is, for example, the ideally flat surface, by comparison with which any visible thing to which we apply the term "flat surface," salt pan or croquet lawn or billiard table, is to some degree rough. We cannot lightly deny the existence of ideal objects of this kind. Geometry teems with talk of

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such things as ideally flat surfaces; and we have our work cut out if we want to maintain that there are no such things, on the grounds that no visible thing is quite like the planes of geometry. We would have to find some alternative way of recording the difference between flat and rough that we could otherwise record by saying that there is such a thing as the ideally flat surface, whereas there is no such thing as the ideally rough one, a surface so rough that none could possibly be rougher, a surface by comparison with which all visible rough surfaces are flat. Moreover, we would have to find some explanation for why geometry should seem so impressive a branch of knowledge if the objects of which it speaks are no more real than those spoken of in fairy tales.[3] The ideal objects of the mathematicized sciences are therefore no less well established than ideal artifacts. But, unlike ideal artifacts, they are not to be explained teleologically. Or so, at least, it seems.

Plato suggests that appearances here mislead: the way that we usually do geometry is imperfect, and because of this imperfection, we fail to appreciate that geometrical objects too must be seen in the light of the Good. Plato makes this suggestion as part of a string of comparisons, describing both the internal structure of the intelligible realm that is governed by the Good, and its connection with the visible realm that is governed by the Sun. In each case, the description is by a comparison with the internal structure of the visible realm. This is because the descriptions are addressed to people like us, people who, for the moment at least, feel more at home in the visible realm than elsewhere.

Within the visible realm, Socrates points out, Glaucon may observe a contrast between, on the one hand, solid objects such as plants, animals, and artifacts, and, on the other, their shadows and reflections "and everything else of that sort" (509d–510a). These days, we have mirrors of silvered glass that are of a far higher quality than the polished bronze mirrors of antiquity. In consequence, we are accustomed to think of some reflections ("mirror images") as exact counterparts of their originals, differing only by the interchange of left with right. We need to forget such thoughts if we are to appreciate Socrates' comparisons. We need instead to join the ancients, and think of reflections generally as partial and distorted likenesses that

[3] For the start of such an explanation, see Quine 1960, pp. 248–57.

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are – the phrase is hard to resist – no more than shadows of their originals.[4] This contrast between solids and their shadowy semblances is, continues Socrates, correlated with a contrast between two kinds of mental state or event (511d): "trust" (*pistis*), and "fancy" (*eikasia*). The contrast is, we are told, that trust has more clarity (*saphēneia*) than fancy, just as the solids that are the objects of trust have more truth (*alētheia*) than the shadows and reflections that are the objects of fancy (511e; cf. 509d, 510a). The idea is not that absolutely every sighting of a shadow or a reflection amounts to no more than fancy, as if, merely by observing the earth's shadow during a lunar eclipse, astronomers fall into the most degraded of cognitive states. The idea is rather that fancy is the state of mind in which one is looking at some shadow or reflection, without appreciating that it is only a shadow or reflection. It is the state of mind of a kitten whom I once saw arch her back, fluff her fur, and hiss and spit at her own image in a mirror: she took the image to be another kitten, and responded accordingly.[5] Trust, by contrast, is the state of mind of one who appreciates that shadows and reflections are only shadows and reflections, imperfect likenesses of solid objects, but who makes about the visible realm the sort of mistake that the kitten made about her image in the mirror, the mistake of not appreciating that the visible realm itself is only an imperfect likeness of the intelligible realm that is governed by the Good.

Fancy and trust amount collectively to "opinion" (*doxa*), and as fancy is related to trust, so too, says Socrates, opinion as a whole is related to "knowledge" (*noēsis* is the term used in 534a; 510a suggests that Plato would also be happy with the term *gnōsis*): opinion is less clear than knowledge, and the objects of opinion are less true than the objects of knowledge (509d–510a, 534a).

We can understand this contrast between knowledge and opinion if we recollect the classic proof of the incommensurability of the

[4] See, e.g., Aeschylus, *Agamemnon* 839; Sophocles, *Ajax* 126, *Philoctetes* 946–47, for other ancient talk of reflections and shadows in the same breath.

[5] Don't just laugh at the kitten. Her state may well be yours too, if the analogy of the cave is right to suggest that *eikasia* is or is analogous to something that pervades the life of all human beings from childhood onward, unless and until they receive a proper education (514a–515b). For more on what this pervasive *eikasia* might be, see section I of David Sedley's chapter 10 in this volume.

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diagonal of a square with its side.[6] Assume, for the moment, that the diagonal of a square is commensurable with its side; assume, in other words, that the side can be divided into exactly S equal units, and that the diagonal is exactly D of those units. If your choice of unit makes both S and D even, then double the length of the unit, and keep doubling it until at least one of them is odd. Now since, as we can remind ourselves by reading *Meno* 82b–85b, the square on the diagonal of a given square is double the area of the original square, $D \times D = 2(S \times S)$. So $D \times D$ is even. And so D is even too; in which case, since we have so chosen our unit that D and S are not both even, S would have to be odd. Now since D is even, there is an E such that $D = 2E$. So $4(E \times E) = 2(S \times S)$. So $2(E \times E) = S \times S$. So $S \times S$ is even. So the odd number S is also even. Which is absurd. So the diagonal of a square cannot be, as we earlier assumed, commensurable with its side.

The knowledge that we get from this proof has a clarity that we could never achieve from a perceptible square. Even the daintiest techniques of measuring perceptible squares will have their margin of error. They cannot make us certain of anything precise. A technique accurate to within one part in a thousand cannot make us certain of anything more precise than that the diagonal is somewhere between 1,413 and 1,415 thousandths of the side. And being certain of such a measurement will leave entirely unsettled many questions settled definitively by our mathematical proof, such as whether the diagonal of a square is or is not exactly 1,414,213 millionths of the side. Moreover this lack of clarity in the opinions that we can form about a perceptible square corresponds to a deficiency in perceptible squares themselves. No two sides of a perceptible square are exactly 1,000,000 millionths of one another; indeed, not even one side is so precisely bounded as to have an utterly determinate length. This deficiency in perceptible squares means that they are not truly squares, any more than the kitten in the mirror was truly a kitten. Because of this deficiency, our knowledge that the diagonal of a square is incommensurable with its side is not, in any obvious way, knowledge about perceptible squares. The natural inference is

[6] See Euclid, *Elements* Book 10, Appendix 27. This proof is apparently alluded to in 510d; certainly, 534d and 546c allude to related results about incommensurability.

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that it can only be knowledge about something else, whose degree of truth matches the degree of clarity of the knowledge.

Furthermore, adds Socrates, there is, within knowledge, a distinction that corresponds to the distinction within opinion between fancy and trust. For knowledge has the two subspecies of "thought" and "intellect" (*dianoia* and *noēsis* in 511c–e, *dianoia* and *epistēmē* in 533e); and thought is, in respect of clarity and truth, inferior to intellect, as fancy was inferior in those respects to trust. It is in the region of thought that we will be able to locate mathematics and its objects, or – to be more precise – the imperfect sort of mathematics that does not appreciate the dependence of its objects on the Good.

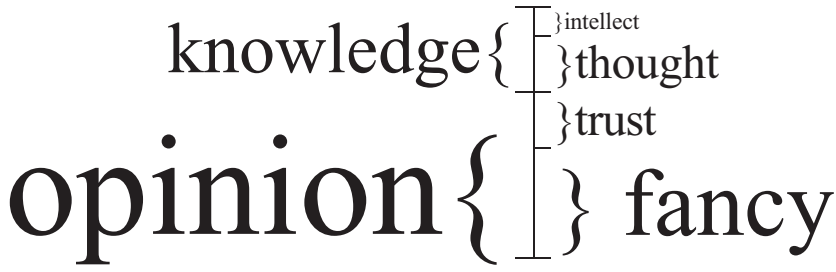
Socrates sums up these comparisons in a mathematical object, the divided line. Divide a line, he instructs Glaucon, into two segments of unequal length. One segment represents opinion, the other represents knowledge, and their difference in length represents the way that opinion and knowledge differ in clarity, and their associated objects differ in truth. Then, in order to represent fancy, trust, thought, and intellect, their relative clarity, and the relative truth of their associated objects, subdivide each of these two segments into two subsegments, one of which stands to the other in the same ratio as the two segments of the original line (509d; Euclid *Elements* Book 6, Proposition 10 gives a procedure for making such divisions).

In obeying these instructions about ratios, Glaucon is faced with various arbitrary choices. For example, is he to make the line horizontal or not? And which subsegment is he to place at which end? As it happens, the line that he constructs is vertical, and its four subsegments for intellect, thought, trust, and fancy come in that order from top to bottom (511d–e, 533e–534a). Thus Glaucon might have constructed his line as in this diagram (the divided line according to Proclus):



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But he might instead have constructed it as in this diagram (the divided line according to Plutarch):



For when we obey Socrates' instruction to have the segment for opinion unequal to the segment for knowledge, we have to make also an arbitrary choice about which is to be the longer.

Some commentators claim that the choice here is not arbitrary. Proclus (*Commentaries on the* Republic *of Plato* 1.289.6–18 Kroll) says that a greater length represents a greater clarity, on the grounds that the intelligible is superior to and encompasses the visible and that what encompasses something is always greater than what it encompasses; while Plutarch (*Platonic Questions* 1001d–e) says that a greater length represents a greater obscurity, on the grounds that the divisibility, indefiniteness, and multiplicity of what is visible should be represented by a greater length. This wrangle between Plutarch and Proclus serves only to confirm the arbitrariness of our choice.

Why have us make such an arbitrary choice? When we draw a diagram in obedience to Socrates' instructions, we are producing a visible representation. Such a representation inevitably contains irrelevant features, features that correspond to nothing in the reality represented, features that in consequence cannot be explained by reference to the purpose of the representation. The proper response to such features is of course to ignore them, to abstract from irrelevant clutter, and think beyond the representation to the reality that it represents. Sometimes, such things are easily done: we all find it easy to realize that a foot-long picture of a snake need not be a picture of a foot-long snake. And it is, of course, because of our universal ease with interpreting visible images of visible things that Plato uses our

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dealings with such images to expound to us the intelligible realm. It is however not so easy to abstract from a diagram that we have drawn in obedience to Socrates' instructions. And this is not merely because we will have to abstract from the visible diagram to a geometrical object, a perfectly straight line, of no breadth, of no particular color, bounded and divided by perfectly sharp points. For even then, we will have to abstract further. We will need to realize that, however striking the difference between a line that has the opinion segment half the length of the knowledge segment, and one that has it forty-seven times the length instead, nevertheless, this difference too is altogether insignificant. We must abstract even from these mathematical features, in just the way that we had to abstract from the smudginess of the lines and points in our original diagram. We need to ignore everything about the diagram, save only the fact that it represents segments and subsegments related to one another in the complex series of proportions specified by Socrates. When we treat in this way a diagram that we have drawn, we are getting some practice in the kind of strenuously abstract thought that is needed to take us from the visible to the intelligible realm. And to give us such practice was no doubt the purpose of having us choose arbitrarily between making the opinion segment longer than the knowledge segment and making it shorter instead. The wrangle between Proclus and Plutarch shows how much we need that practice.

The divided line that we represent in our visible diagram has one surprising feature. Without ever affirming so outright, Plato leads us to expect that the four subsegments of the line, the subsegments for intellect, thought, trust, and fancy, will each differ in length. For the fact that they are arranged in that order from top to bottom of the line (511d–e, 533e–534a) seems to correlate with an order of merit among the mental states that they represent, and we are not told of any differences in merit other than those indicated by differences in length. Indeed, when Socrates says that geometers treat as images of geometrical objects those things that are imitated by shadows and reflections (510b, 510d–e, 511a), he comes close to implying not only that thought and trust have unequal subsegments, but also that the subsegment for thought has to that for trust the ratio that the subsegment for trust has to that for fancy. In fact, however, the subsegments for thought and for trust will turn out to be equal in length. And this

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is inevitably so, no matter how the line is divided, so long as it is divided as Socrates instructs.[7]

This surprising equality between the subsegments for trust and for intellect can hardly be accidental. Plato associated with two men who could have taught him far greater subtleties about the theory of proportion: Archytas of Tarentum, a friend (Plato *Epist.* 338c–d) whose remark that the various mathematicized sciences are all sisters (DK 47 B 1) is alluded to with approval in 530d (cf. 511b), and Eudoxus of Cnidus, a "pupil" (Cicero *De Divinatione* 2.42.87, *De Republica* 1.14.22) whose theory of planetary motions was the first great achievement in the mathematicized astronomy programmatically described at 528b–530c. Archytas showed how, given two straight lines A and B, we may construct two more straight lines X and Y, so that the ratios of A to X, of X to Y, and of Y to B are all equal; he showed also that, if A, B, and C are positive whole numbers such that the ratio of A to B is the same as that of C to C + 1, then there is no positive whole number X such that the ratio of A to X is the same as that of X to B.[8] Eudoxus originated the general theory of proportion expounded in Euclid *Elements* Book 5 (see Book V, Scholia 1 and 3): earlier thinkers had proved only piecemeal, by one proof for proportions among lines, another for proportions among positive whole numbers, and so on, results that Eudoxus was able to formulate and prove quite generally, for proportions among magnitudes of an arbitrary kind (cf. Aristotle *Posterior Analytics* 74a17–25). And there is direct evidence in the *Republic* itself that, whether or not he had learned it from Archytas and Eudoxus, Plato had knowledge of one slight but relevant subtlety within the theory of proportion. For in the passage of recapitulation at 534a, he has Socrates say that "as knowledge is to opinion, so intellect is to trust and thought is to fancy." These things are consequences of what Socrates said earlier, when first explaining the divided line, but not obviously so. They are much less obviously consequences than, for example, "as knowledge is to opinion, so intellect is to thought and trust is to fancy." And anyone who makes one of his characters infer the less obvious

[7] Exercise for the reader: prove this using the techniques of the theory of proportion set out in Euclid, *Elements* Book 5.

[8] DK 47 A 14, and DK 47 A 19. English translations of these passages can be found in Thomas 1939, pp. 131–33, 285–90.

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consequence that "as knowledge is to opinion, so intellect is to trust" would have little difficulty in himself inferring the more obvious consequence that "as knowledge is to opinion, so intellect is to thought," and then putting these two together and himself inferring that thought is equal to trust. It is therefore hard to doubt that Plato chose the image of the divided line in full awareness that its subsegment for thought is inevitably equal to its subsegment for trust.

There are several reasons why Plato might have made such a choice. On the one hand, if the divided line does have the immediate purpose of expounding the superiority of thought over trust, then its very defectiveness for that immediate purpose would fit it to serve Plato's purposes in the longer term. It is central to Plato's thought about images that an image always falls short of the original of which it is an image. That is why he finds it so appropriate to describe the relationship between perceptible and intelligible as a relationship between image and original. The divided line is itself another image. So if it too is defective, that would be all to the good. On the other hand, perhaps the divided line is meant instead to hint that, in itself, thought is not after all superior to trust, that if we make the transition from trust to thought, but then go no further, we have not in fact improved ourselves cognitively, and that thought is superior to trust only in that thought is adjacent to, whereas trust is one step further removed from, the finest of all cognitive states, intellect. In this case, the fact that the divided line inevitably has equal subsegments to represent both trust and thought would be no defect at all. Finally, it may be that the divided line was chosen because it allows both these incompatible interpretations. For if images were not puzzlingly contradictory, then we would be liable to rest content with images, rather than be provoked to go beyond them to the reality from which they derive. Recall the discussion of fingers and size in 523b–524d: the senses unambiguously report fingers as fingers, and that does nothing to provoke the mind to think about what a finger is; but the mind is provoked to wonder what big and small might be, when the senses report that a single finger is both these things.

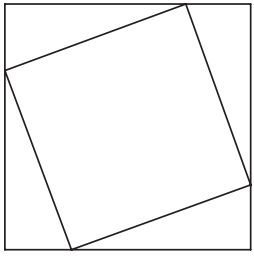
When you are engaged in thought, you treat physical objects as images of an intelligible reality. You appreciate that they are derivative and inferior copies of something more real and more clear than themselves, and you use them only as a device to help you think about the reality that they represent. Your cognitive state is therefore

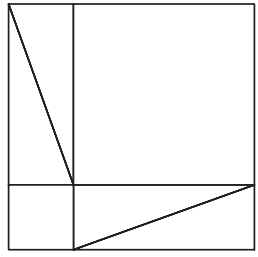
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superior to mine, at my first encounter with geometry. My primary school teacher had told us of Pythagoras' Theorem. I therefore set out to test the theorem, by cutting a right-angled triangle out of colored card, and measuring its sides. I then did my sums, and concluded, with disappointment and indignation, that Pythagoras was wrong. But it was of course I who was wrong. Any mismatch between Pythagoras' Theorem and my right-angled triangle was to be blamed on my triangle, not on the theorem. For the right-angled triangle that I had cut from colored card was, in a crucial sense, no more a right-angled triangle than the kitten in the mirror was a kitten. Geometrizing about the bit of colored card was therefore as mistaken as hissing and spitting at the reflection. But, once you realize that a reflection is only a reflection, you may let it guide you in your dealings with what it reflects: you see that your image in the mirror has untidy hair, and you may use this to help you in tidying your own. Likewise, provided that you appreciate the difference between bits of colored card and the shapes spoken of in geometry, you may use the former to guide your thought about the latter.

Using physical objects as images of an intelligible reality is both a merit and a defect in the kind of intellectual activity that we have been labeling "thought." It is a merit, in that it makes thought superior to trust, which has no notion that physical objects are only images. It is a defect, in that it makes thought inferior to intellect, which has no need of such images (510b, 511c). And it is not the only defect in thought. Besides using physical objects as images, thought also, complains Socrates, conducts its enquiries on the basis of hypotheses, without being able to go beyond them to anything higher (510c, 511a, 511c–d). What exactly is this complaint? And how does thought's use of hypotheses relate to its use of images?

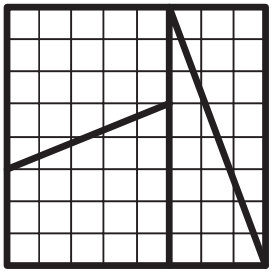
Look at this pair of diagrams:

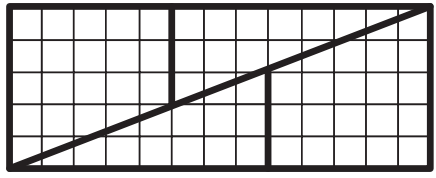




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and consider how they are used in this proof of Pythagoras' Theorem: "The big square on the left is formed by fitting four copies of a right-angled triangle together with the square on its hypotenuse. The big square on the right is formed by fitting another four copies of the same right-angled triangle together with the squares on its other two sides. The big square on the left is equal to the big square on the right. But equals taken from equals leaves equals. So take away the four copies of the right-angled triangle from each of the big squares, and you will still be left with equals. So the square on the hypotenuse is equal to the sum of the squares on the other two sides." In our proof, we make all sorts of assumptions about how shapes can be fitted together to form other shapes, for no better reason than that these assumptions are suggested by the diagrams. But making such assumptions, for such a reason, is a hazardous enterprise. For look at this pair of diagrams (Lewis Carroll's Theorem from Fisher 1975, p. 92):



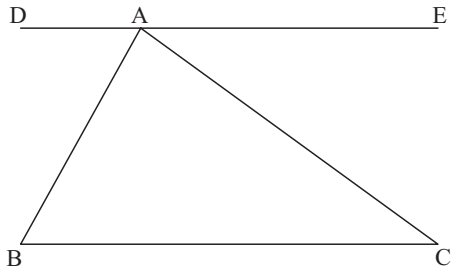


and consider this piece of reasoning that relies on assumptions suggested by these diagrams: "The big square on the left is formed by fitting two copies of a right-angled triangle together with two copies of a lopsided quadrilateral. The big rectangle on the right is formed by fitting another two copies of that same right-angled triangle together with another two copies of that same lopsided quadrilateral. But equals added to equals gives equals. So the eight-by-eight square on the left is equal to the five-by-thirteen rectangle on the right. So sixty-four is equal to sixty-five."

Both pieces of reasoning make apparently similar appeals to diagrams. The second piece of reasoning has evidently gone wrong somewhere. In consequence, we cannot treat the first piece of reasoning, in its current state, as a rigorous proof of Pythagoras' Theorem.

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Nevertheless, the two pieces of reasoning are not equally bad. We can distinguish between them by subjecting them to the further discussions that, in *Meno* 85c–d, Socrates suggests for someone who has just been persuaded by a diagram of a special case of Pythagoras' Theorem. We can do little to elaborate on our argument that the eight-by-eight square is equal to the five-by-thirteen rectangle. We will soon get bogged down in muddle and contradiction if we attempt to argue that the pieces into which the square is cut can be assembled into a rectangle without the slightest gap on what looks to be its diagonal. We can however elaborate on the argument for Pythagoras' Theorem, to say much more about how the various shapes in the diagrams really do fit neatly together, just as they seem to, without any funny business by way of stretching or shrinking or gaps or overlaps or bulges or indentations. For example, we can show that right-angled triangles and regular quadrilaterals really do fit together, just as our diagrams suggest, once we have shown that each interior angle of a regular quadrilateral is a right angle, and that in any triangle containing a right angle the other two angles sum to one right angle. We can show these things, once we have shown that the interior angles of a triangle sum to two right angles. And we can argue for this in its turn by, for example, this argument that Eudemus ascribed to "the Pythagoreans," and that was therefore presumably already familiar to Plato. The Pythagoreans' proof comes from Proclus, *Commentary on Book 1 of the* Elements *of Euclid*, 379.2–16:



"Let ABC be a triangle, and let DE be drawn through A parallel to BC. So since BC and DE are parallel, the alternate angles are equal. So the angle DAB is equal to the angle ABC and the angle EAC is equal to the angle ACB. Let the angle BAC be added in common. So the angles DAB, BAC and CAE, that is, the angles DAB and BAE, that is, two

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right angles, are equal to the three angles in the triangle ABC. So the three angles of the triangle are equal to two right angles." We might even set out our entire argument in the axiomatic fashion familiar to us from the *Elements* of Euclid, and presumably familiar to Plato from the now lost books of *Elements* by his predecessor Hippocrates, and by his colleagues in the Academy Leon and Theudius (Proclus, *Commentary on Book 1 of the* Elements *of Euclid* 66.7–8, 66.20–21, 67.12–15). We can thus elaborate our earlier argument for Pythagoras' Theorem into something that will better deserve the name of "proof," for it will avoid reckless reliance on diagrams.

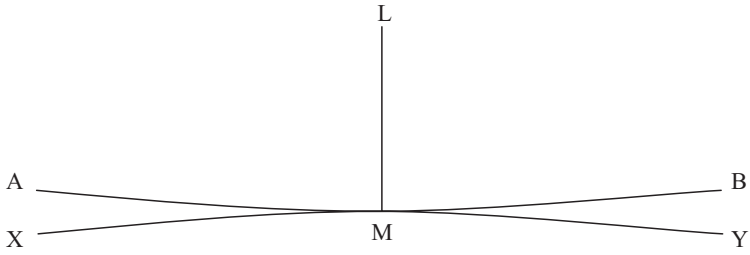
Nevertheless, avoiding reckless reliance on diagrams is not the same as avoiding all reliance on diagrams. For consider the Pythagoreans' proof that the angles of a triangle sum to two right angles: its instruction "let DE be drawn through A parallel to BC" is futile unless there is a line through A parallel to BC; and its inference "since BC and DE are parallel, the alternate angles are equal" is wrong if there is more than one line parallel to BC that passes through A. The proof then relies on the assumption that for any straight line and any point not on that line, exactly one parallel to that line passes through that point. And any proof of Pythagoras' Theorem will have to rely, if not on this assumption so formulated, at least on some equivalent thereof. But why accept this assumption? People did attempt to derive it from more basic geometrical principles. However, as Aristotle in *Prior Analytics* 65a1–9 observed, all their attempted derivations unwittingly argued in circles; and more recent developments have shown why no better argument is possible.[9] We therefore fall back on the diagram. This perhaps confirms our assumption, in that, for example, we can't add to the diagram a line that looks straight, that looks parallel to BC, that looks to pass through A, and that also looks distinct from DE. But this confirmation is a curious one. For the diagram would not look any different if several lines through A were parallel to BC, and all of them were, for a few million miles on either side of A, too close together for perception to tell them apart. Nor would the diagram look any different if every straight line through A met BC sooner or later, in some cases so much later that, for a few million miles on either side of A, the line through A did not get perceptibly closer to BC. So the diagram

[9] For a fairly untechnical account, see Sklar 1997, pp. 13–25.

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confirms our assumption that there is a parallel to BC through A only in that DE does not look as if it has to meet BC sooner or later. Moreover, in giving us this curious confirmation that there is exactly one parallel to BC through A, the diagram gives us at most an assurance of this fact; it gives us no explanation. It tells us, at best, that this is so; it does not begin to tell us why.

Or again, consider how, in our elaborated proof of Pythagoras' Theorem, we would assume that each angle belongs to just one of three entirely distinct kinds: either it is a right angle, or it is smaller than some right angle and so acute, or it is bigger than some right angle and so obtuse. An alternative version of the assumption would be that "all right angles are equal to one another" (Euclid *Elements*, Book 1, Postulate 4). We would rely on this assumption in reasoning that when four angles are assembled around a point, and each is equal to some right angle, then the four angles fit together snugly. Why make this assumption? A right angle is defined thus: "whenever a straight line standing on a straight line makes the adjacent angles equal to one another, then each of the adjacent angles is right" (Euclid *Elements* Book 1, Definition 10). This definition requires that each right angle is one of a pair of right angles, each equal to the other of the pair; this definition says nothing about whether either is equal to any third right angle. Will diagrams help where the definition does not? Diagrams seem to indicate that all right angles are equal, in that we cannot draw a diagram that makes the alternative look possible. For look at this diagram



and try to imagine that LM is a straight line standing on the straight line AMB, making the adjacent angles AML and LMB equal to one another and therefore right; that LM also stands on the straight line XMY, making the adjacent angles XML and LMY also equal to one

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another and therefore also right; and finally that the right angle XML is bigger than the right angle AML, since it consists of the right angle AML, together with the angle AMX. The obvious retort is that the lines AMB and XMY do not even look straight, and could not look straight if they were in other respects to look as they are imagined to be. And this retort would be perfectly true. But it does nothing to rule out the idea that there are straight lines like AMB and XMY, kissing at a point and elsewhere diverging, but diverging so slowly that, for a few million miles on either side of the point at which they kiss, they look to be the same. And if this idea is correct, then our right angles could be different in size, while still looking all the same. Thus for right angles, as for parallels, we assumed something with only a curious confirmation, and with no explanation at all.

In short, even if we elaborate our argument into the best proof that geometry can manage, we are still like the mathematicians with whom Glaucon was familiar: "when people deal with geometry and arithmetic and the like, they hypothesize odd and even numbers, and the shapes, and three kinds of angle, and related things, depending on the subject. They act as if they know these things, once they have hypothesized them; and do not see fit to give any account of them either to themselves, or to anybody else, as if these things were obvious to everybody. Instead, they treat them as starting points, and go through the subsequent steps until they conclude, by consensus, at the thing they have set out to investigate" (510c–d). As Socrates asks at 533c: "How on earth could this sort of consensus ever amount to understanding?" Can we do any better than geometrical thought?

According to the divided line, geometrical thought is analogous to fancy, and we can do better than fancy by appreciating that reflections and shadows and the like – the objects that fancy accepts uncritically – are in fact only reflections and shadows, derivative from the three-dimensional visible objects that are their originals. We should therefore be able to do better than geometrical thought by appreciating that what it hypothesises is in fact only derivative from something else. But what exactly does geometry hypothesize? And what is the something else from which that is derivative?

According to Aristotle *Metaphysics* 987b14–18 (cf. 1028b19–21), Plato "says that, in addition to perceptible objects and Forms, things

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include also mathematical objects: they are intermediate, distinct from the perceptibles in that they are eternal and unchangeable, and distinct from the Forms in that there are several mathematicals of a kind, whereas each Form is unique." The idea is, for example, that since there are three straight lines in a triangle, then, if there is even one perfect triangle, there must be several perfectly straight lines; and so there must be other straight lines besides the perceptible ones (which are imperfect) and the Form of the straight line itself (which is unique). In what we have of Plato, he never says this in quite so many words. But he does come close to doing so in our passage from the *Republic*. At 511d, Glaucon describes mathematical thought as "something intermediate between opinion and intellect." Because mathematical thought does not have quite the clarity of intellect, the objects of mathematical thought, intelligible though they are, cannot have quite the truth of the objects of intellect (511e). Intellect and mathematical thought must therefore have distinct objects. Since the objects of intellect are "Forms themselves" (510b), the objects of mathematical thought must therefore be other than Forms. Nevertheless, the objects of mathematical thought will be as eternal and unchangeable as any Form: geometers talk of "squaring" and "applying" and "adding," as if they are engaged in activities; but they are well aware that the aim of geometry is "knowledge of what exists forever, not of what at some particular time comes into and goes out of existence" (Socrates at 527a–b; Glaucon, less cautiously, says at 527b that geometry is such knowledge, and not merely that it has such knowledge for its aim). As for the thought that "there are several mathematicals of a kind," this has its counterpart at 526a: the mathematicians are asked, "What kind of number can you be talking about, where the unit meets your requirements: each equal, every one to every one, not different in the slightest, and containing no part within it?"; and they reply that they are talking about intelligible numbers. The idea is that mathematicians might illustrate the mathematical truth that two threes make six by laying out two rows each of three pebbles; they will ignore as irrelevant the fact that some of the pebbles are rounder than others, and the fact that each one of the pebbles is also two half-pebbles; that is, they will pretend that each of the six pebbles is a perfect unit, perfectly interchangeable with any other; and they will treat the mathematical truth illustrated by the pebbles as a truth about pairs of trios, where

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the many units actually are as the mathematicians only pretended the pebbles to be.[10]

Against all these indications that the divided line represents mathematical thought as dealing with mathematical intermediates, some scholars set 510d–e, which describes mathematicians as "presenting their arguments with a view to the square itself, and to a diagonal itself, rather than to the one that they draw." This, some say, means that mathematical thought deals not with intermediates but with Forms.[11] Certainly, such "so-and-so itself" locutions can be used to indicate Forms. But that is not their only use. Thus when 363a speaks of people who fail to praise justice itself, the point is not that they fail to praise the Form of justice, but that they commend justice, not for any intrinsic merits, but for the good reputation that can be its consequence; when 394c speaks of stories told by the utterances of the poet himself, it does not mean the works of the Form of poet, but stories narrated by a poet speaking entirely *in propria persona*, without any of the speeches that form part of epic and the whole of drama; when 404c speaks of cooking things with fire itself, it does not mean cooking them with the Form of fire, but barbecuing them, by contrast with cooking them in pots; when 454a speaks of people wrangling over the name itself, the point is not that they wrangle over the Form of name, but that they wrangle just over words, rather than over what they mean; when 525e speaks of the unit itself, it means, not the Form of unit, but the multiple units, "each equal, every one to every one, not different in the slightest, and containing no part within it," that compose the intelligible numbers studied by mathematicians, as contrasted with the divisible and heterogeneous things that stand in for units in mathematicians' visual aids; when 604a speaks of how the experience itself can make us grieve, it does not mean the Form of experience, but particular experiences in the mind of someone who is not sufficiently rational and law-abiding to resist them. In general, then, Plato uses "so-and-so itself" locutions for any so-and-so free of some kind of clutter (consequences, actors, cooking pots, or whatever) that is indicated in the context;

[10] The idea may now look grotesque, but it has looked so only since Frege 1978 [1884], ch. 3. Not even this immediately persuaded every mathematician: see Frege 1979, pp. 72–86, for an entertaining restatement, provoked by work published in 1887, "as though the third section of my *Grundlagen* had never been written."

[11] See, e.g., Robinson 1953, p. 197, and Annas 1981, p. 251.

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and when he uses such a locution for the Form of so-and-so, that is only because the Form is an uncluttered so-and-so. Hence, when mathematicians are described as "presenting their arguments with a view to the square itself and to a diagonal itself, rather than to the one that they draw," this means only that they are talking about geometrical objects free of something that clutters their diagram, free of what makes the diagram only an imperfect illustration of the fact that the diagonal of a square is incommensurable with its side. But "mathematical intermediates" would be such objects. We may therefore accept the indications that mathematical thought is concerned neither with perceptibles nor with Forms, but with things intermediate between the two. We may therefore accept also that, when we go beyond mathematical thought to intellect, we come to see mathematical intermediates as derivative from Forms, and to focus our minds directly on Forms themselves.

But why should intellect be an improvement on thought? What is it about Forms that means that, when thinking about them, our reasoning can be something other than just yet more deductions from peremptory and unexplained assumptions? According to Plato, intellect reasons dialectically (511b–c, 533c–535a), and proficiency in dialectic requires a maturity, and a prior education, that even under the best of circumstances few will have (536d–540b). So we should not expect it to be easy to understand why intellect is an improvement on thought. Let us do what we can.

It is easy enough to take a generalization about mathematical intermediates of some kind and derive it from the corresponding singular statement about the corresponding Form. For example, "In any isosceles triangle, the two equal sides subtend two equal angles" can be derived from "The ideal isosceles triangle has two equal sides, subtending two equal angles." We can do this also with generalizations that geometers would treat as the unexplained starting points of their proofs. Thus Euclid, in *Elements* Book 1, Definition 22, defines a square as "Among quadrilateral figures, one that is both equilateral and right-angled is a square." This is trivially derivable from "The ideal square is a figure with right angles and four straight and equal sides." Again, Euclid, in *Elements* Book 1, Postulate 4, postulates "All right angles are equal to one another." It is only slightly less trivial to derive this from "The ideal right angle is the universal standard by which angles are to be measured." Such derivations do not,

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in themselves, mark any great advance. Even if they explain what geometers can only hypothesize, they do so by affirming things that are no less in need of explanation. If such derivations are any advance, that is for two reasons. First, their affirmations about Forms are so tentative that we will see them as the hypotheses that they are, and not treat them as underivative and self-explanatory starting points (511b). Second, we can work our way up from these hypotheses to something that genuinely is an unhypothetical and self-explanatory starting point of the lot (510b, 511b).

Given what we were told in the image of the Sun, the unhypothetical starting point can only be the Good. But how might the Good be the unhypothetical starting point of mathematics? Plato, we may hope, does not envisage a novel axiomatization, where from axioms about the Good we deduce theorems about mathematics, and include among those theorems mathematical propositions that are currently taken as axiomatic. For if the novel axioms told us only about the Good, and did not also mention either squares or triangles, then we could deduce from them no conclusion about the triangle that would not apply just as well to the square; while if the novel axioms told us not only about the Good, but also about squares and triangles and all their distinctive features, then these axioms would hardly give the Good its unique place as the unhypothetical starting point of all the rest. And in any case, why should presenting mathematics by deduction from axioms that describe the Good, whether as well as or instead of "odd and even numbers, and the shapes, and three kinds of angle" (510c), be any more satisfactory than presenting mathematics in the style with which Glaucon was familiar? After all, Glaucon's dissatisfaction at 510c–d comes close to a dissatisfaction with the axiomatic style generally, and its habit of giving no further explanation of what seems so obviously true that it can be asserted without proof. Glaucon's dissatisfaction with mathematics as customarily presented would therefore be resolved not by replacing customary axioms with new ones that mention the Good but by supplementing customary axioms with some explanation of the facts that they so peremptorily assert.

Plato's idea seems to be that the Good will provide such an explanation and that we are to explain the existence and character of mathematical Forms teleologically, in something like the way that we explained teleologically the shape of the ideal wheel and the position

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of its axle. For there is an obvious sense in which the Good is the unhypothetical starting point of all teleological explanation: if something is explained as being there in order to achieve some goal, then the explanation is correct only to the extent that achieving the goal is good, and the explanation is completed when, and only when, we appreciate that and how the goal is good. For example, if you say that the wheels of your car have some tread on their tires in order to increase friction, and some oil on their bearings in order to reduce friction, then both explanations can be correct, but only because it is good to have lots of friction between a wheel and the road, and good also to have little friction between a wheel and its axle; and once we appreciate what is good about those various amounts of friction in those various contexts, we will have finished our teleological explanations of the tread and of the oil. In short, having a teleological explanation of something means seeing what is good about it; and it will be for this reason that the Good is the unhypothetical starting point of all that is teleologically explicable. Hence, if mathematical Forms are, like ideal artifacts, teleologically explicable in their entirety, then the Good will be the unhypothetical starting point of the whole of mathematics.

So how might we give a teleological explanation of mathematical Forms? The idea seems to be that mathematical Forms are as they are because of a certain sort of orderly beauty that this makes possible. Suppose, for example, that the ideal square is not right-angled. To be more precise, and to avoid quibbles that a square is right-angled "by definition," suppose that the ideally regular (i.e., equilateral and equiangular) quadrilateral has for its interior angles something other than right angles. (This is tantamount to denying the proposition about parallels that lay behind the Pythagoreans' proof that the interior angles of a triangle sum to two right angles.) In that case, it will not be possible to assemble four equal regular quadrilaterals to make a larger regular quadrilateral. Indeed, quite generally, there will be no two figures identical in shape but different in size, and space will not be uniform from one scale to another. This would be like the idea scorned in *Timaeus* 62c–63a (cf. *Phaedo* 108e–109a) that space is not uniform in different directions and that there is a cosmically privileged downward: if either idea were right, there would be an ugly arbitrariness to the world. Or again, suppose that the ideal right angle does not provide a universal standard

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by which angles are to be measured and that two right angles can be of different sizes, just as two obtuse or two acute angles can. On this supposition too, space would not be uniform. It would vary from one region to another: there could be two right-angled triangles, each with its right angle enclosed by sides of unit length, that could not be superimposed. The world would be messy and disordered, much as *Timaeus* 37c–39e suggests that the world would be a mess without the universal standard for the measurement of time provided by the regular motions of the heavenly bodies. None of this shows that these suppositions are self-contradictory. Nor could such teleological considerations demonstrate in any other way that these suppositions are mistaken. For sometimes, as 519d–e illustrates, one good thing must be forgone if another is to be secured; and it might turn out that the truth of these ugly suppositions is the price that must be paid if the Forms are to be, all things considered, as good as they can be. (The fact that teleological considerations can be set against one another in this way is of course one of the things that distinguishes teleological explanation from deduction; for a valid deduction can never be invalidated by adding new premises, not even if they contradict the old.) These teleological considerations do, however, show that these suppositions offend our sense of beauty, order, and even perhaps fairness. If we are inclined to reject these suppositions on such grounds, that is a modest venture into the upmost part of the divided line, where we see Forms in the light of the Good.[12]

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