# Optimization B – QEM1 and IMMAEF Track 2 Slides 1 - One week

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# Outline

- Optimization problems : Basic notions
- Existence of a solution
- Uniqueness of the solution
- Unconstrained optimization
- Equality constrained optimization
- About the Lagrange multipliers

## Textbooks

Sydsaeter K., Hammmond P., Seierstadt A., Strom A. (2005) : Further Mathematics for Economic Analysis, Prentice Hall.

Sundaram R.K. (1999) : A First Course in Optimization Theory, Cambridge University Press, Cambridge.

# Basic notions

$D$ is a subset of $\mathbb{R}^n$ with $n \geq 1$. Let $f$ be a function from $D$ to $\mathbb{R}$ :

$$f : x \in D \subseteq \mathbb{R}^n \to f(x) \in \mathbb{R}.$$

Let $S$ be a subset of the domain $D$. The set $S$ is the set of **feasible points** (or **admissible points**), and sometimes it can be described by a finite number of constraints.

An **optimization problem** consists in finding the maximum (respectively, the minimum) value of $f$ on the set $S$. It is denoted by :

$$(\mathcal{P}) \max_{x \in S} f(x) \qquad \textit{resp.} \quad (\mathcal{Q}) \min_{x \in S} f(x)$$

In both cases, $f$ is called the **objective function**.

# Solutions and value of an optimization problem

## Definition

- The point $\overline{x}$ is a **solution** of problem $(\mathcal{P})$ if $\overline{x} \in S$ and for all $x$ in $S$ :

$$f(x) \leq f(\overline{x}).$$

- The point $\overline{x}$ is a **solution** of problem $(\mathcal{Q})$ if $\overline{x} \in S$ and for all $x$ in $S$ :

$$f(\overline{x}) \leq f(x).$$

- $\mathcal{S}ol(\mathcal{P})$ denotes the set of solutions of problem $(\mathcal{P})$, and $\mathcal{S}ol(\mathcal{Q})$ denotes the set of solutions of problem $(\mathcal{Q})$.

## Definition

- The value of problem $(\mathcal{P})$ is the supremum of the set $\{f(x) \mid x \in S\}$.
- The value of problem $(\mathcal{Q})$ is the infimum of the set $\{f(x) \mid x \in S\}$.

If $\overline{x}$ is a solution of problem $(\mathcal{P})$, then $f(\overline{x}) = \max\{f(x) \mid x \in S\}$ and $f(\overline{x})$ is called **the maximum value** of $f$ on $S$.

If $\overline{x}$ is a solution of problem $(\mathcal{Q})$, then $f(\overline{x}) = \min\{f(x) \mid x \in S\}$ and $f(\overline{x})$ is called **the minimum value** of $f$ on $S$.

**Remark**

*If problem $(\mathcal{P})$ (respectively, problem $(\mathcal{Q})$) has **several solutions**, for instance, $\overline{x} \in S$ and $\widetilde{x} \in S$ with $\overline{x} \neq \widetilde{x}$, then it must be that :*

$$f(\overline{x}) = f(\widetilde{x}).$$

*That is, **the maximum value** (respectively, **the minimum value**) of f on S is **unique**. This is a simple consequence of the definition of solution.*

# Local solutions

Let $d : D \times D \to \mathbb{R}_+$ be a distance on $D \subseteq \mathbb{R}^n$, for instance, the **Euclidean distance**. The **open ball** in $D$ of center $\bar{x}$ and radius $r > 0$ is :

$$B(\bar{x}, r) = \{x \in D \mid d(x, \bar{x}) < r\}.$$

## Definition

- The point $\overline{x}$ is a **local solution** of problem $(\mathcal{P})$ if $\overline{x} \in S$ and there exists $r > 0$ such that for all $x$ in $S$ with $d(x, \overline{x}) < r$, we have that $f(x) \leq f(\overline{x})$.

- The point $\overline{x}$ is a **local solution** of problem $(\mathcal{Q})$ if $\overline{x} \in S$ and there exists $r > 0$ such that for all $x$ in $S$ with $d(x, \overline{x}) < r$, we have that $f(\overline{x}) \leq f(x)$.

# Examples of optimization problems

**Consumer behavior.**

The utility function $u$ represents the preferences of the consumer on consumption bundles $x = (x_1, ..., x_L) \in \mathbb{R}^L_+$.

Let $p = (p_1, ..., p_L)$ be a price system and let $w \in \mathbb{R}_+$ be the wealth of the consumer. The demand of the consumer is the set of solutions of the following maximization problem :

$$\begin{array}{l} \max u(x_1, ..., x_L) \\ (x_1, ..., x_L) \in S \end{array}$$

where the set of **feasible points** is :

$$S = \left\{ (x_1, ..., x_L) \in \mathbb{R}^L : \left\{ \begin{array}{l} x_\ell \geq 0, \forall \ell = 1, ..., L \\ p_1 x_1 + ... + p_L x_L \leq w \end{array} \right. \right\}$$

## Cost minimization.

We consider a firm that produces the good $L$, by using the goods $1, ..., L-1$ as inputs. Its production function is $f : (z_1, ..., z_{L-1}) \in \mathbb{R}_+^{L-1} \to f(z_1, ..., z_{L-1}) \in \mathbb{R}_+$.

Let $p = (p_1, ..., p_{L-1})$ be the price system of the inputs and let $q \geq 0$ be a level of **output**. The cost minimization problem is :

$$\begin{array}{l} \min p_1 z_1 + \ldots + p_{L-1} z_{L-1} \\ (z_1, ..., z_{L-1}) \in S \end{array}$$

where the set of **feasible points** is :

$$S = \left\{ (z_1, ..., z_{L-1}) \in \mathbb{R}^{L-1} : \left\{ \begin{array}{l} z_\ell \geq 0, \forall \ell = 1, ..., L-1 \\ f(z_1, ..., z_{L-1}) \geq q \end{array} \right. \right\}$$

The demand of inputs of the firm is the set of solutions of the cost minimization problem. The cost function $c(p, q)$ of the firm is the value of the cost minimization problem at $(p, q)$.

**Game theory.**

The best response of a player is the set of solutions of the maximization of his payoff function on the set of his own strategies, by taking as given the strategies of the other players.

Consider a game with two players. For each player $i = 1, 2$, the set $S_i$ denotes the set of strategies of player $i$ and

$$u_i : (s_1, s_2) \in S_1 \times S_2 \to u_i(s_1, s_2) \in \mathbb{R}$$

is the payoff function of player $i$.

The best response of player $i$ for a given strategy $\bar{s}_j \in S_j$ of player $j$, with $j \neq i$, is the set of solutions of the following maximization problem :

$$\begin{aligned} &\max u_i(s_i, \bar{s}_j) \\ &s_i \in S_i \end{aligned}$$

# Extreme Value Theorem (or Weierstrass Theorem)

Let $f$ be a function from $D \subseteq \mathbb{R}^n$ to $\mathbb{R}$. The following theorem provides sufficient conditions for the **existence** of **at least** one solution.

## Theorem

*Problem $(\mathcal{P})$ (respectively, problem $(\mathcal{Q})$) has **at least** a solution if the set of feasible points $S$ is a **non-empty closed and bounded** subset of $\mathbb{R}^n$, and $f$ is **continuous** on $S$.*

• The set $S$ is **closed** in $\mathbb{R}^n$ if and only if it coincides with its **closure** $\mathrm{cl}S$, where :

$$\mathrm{cl}S = \{x \in \mathbb{R}^n \mid \exists\, (x_k)_{k\in\mathbb{N}} \subseteq S \text{ such that } \lim_{k\to\infty} x_k = x\}.$$

• The set $S$ is **bounded** in $\mathbb{R}^n$ if and only if it is included in some closed ball of $\mathbb{R}^n$.

# Uniqueness of the solution

As we have already remarked, an optimization problem might have several solutions.

The following proposition provides sufficient conditions for the **uniqueness** of the solution.

## Proposition

*Assume that the set of feasible points $S$ is **convex** and problem $(\mathcal{P})$ (respectively, problem $(\mathcal{Q})$) has **at least** one solution $\bar{x}$.*

- *if $f$ is **strictly quasi-concave** on $S$, then $\bar{x}$ is the **unique** solution of problem $(\mathcal{P})$.*
- *if $f$ is **strictly quasi-convex** on $S$, then $\bar{x}$ is the **unique** solution of problem $(\mathcal{Q})$.*

# Open sets and interior

Let $U$ be a subset of $\mathbb{R}^n$. The set $U$ is **open** in $\mathbb{R}^n$ if and only if for all $\bar{x} \in U$, there exists an **open ball** $B(\bar{x}, r)$ in $\mathbb{R}^n$ such that :

$$B(\bar{x}, r) \subseteq U.$$

**Proposition**

*a) A finite intersection of open sets is open.*

*b) A union of finitely many or infinitely many open sets is open.*

Let $A$ be a subset of $\mathbb{R}^n$. The **interior** of $A$ is the **largest open** set in $\mathbb{R}^n$ that is included in $A$. The interior of $A$ is denoted by $\text{int}A$. In other words, $\text{int}A$ is the **union** of all the open sets in $\mathbb{R}^n$ included in $A$.

Hence, $A$ is **open** in $\mathbb{R}^n$ if and only if $A = \text{int}A$.

# First order necessary conditions

Let $U$ be an **open** subset of $\mathbb{R}^n$. Let $f$ be function that has all the partial derivatives in every point of $U$. A **stationary point** of $f$ is a point where all the first derivatives are equal to 0.

Consider now the following problem, where we maximize (respectively, minimize) the function $f$ on the **open** set $U$.

$$(\mathcal{P}) \max_{x \in U} f(x) \qquad \textit{resp.} \quad (\mathcal{Q}) \min_{x \in U} f(x)$$

**Theorem**

*If $\bar{x} \in U$ is a **solution** of problem $(\mathcal{P})$ (respectively, of problem $(\mathcal{Q})$), then $\nabla f(\bar{x}) = 0$, i.e.,*

$$\frac{\partial f}{\partial x_i}(\bar{x}) = 0, \ \textit{for all } i = 1, \ldots, n.$$

*That is, $\bar{x}$ is a **stationary point** of $f$.*

# Sketch of the proof

We prove the theorem above for a solution $\bar{x} \in U$ of problem $(\mathcal{P})$. We assume that $f$ is differentiable. Then,

1. the **directional derivative** of $f$ at $\bar{x}$ exists for every direction $h \in \mathbb{R}^n$, $||h|| = 1$, i.e.,

$$D_h f(\bar{x}) = \lim_{t \to 0^+} \frac{f(\bar{x} + th) - f(\bar{x})}{t} \text{ exists and it is finite,}$$

2. $D_h f(\bar{x}) = \nabla f(\bar{x}) \cdot h$.

**Claim 1.** $\nabla f(\bar{x}) \cdot h \leq 0$ for all $h \in \mathbb{R}^n$, $h \neq 0$.

Otherwise, assume that there is a direction $\bar{h} \neq 0$ such that :

$$\nabla f(\bar{x}) \cdot \bar{h} > 0.$$

Then, using the definition of directional derivative, for $t > 0$ sufficiently small, the points $(\bar{x} + t\bar{h})$ belong to some open ball of center $\bar{x}$ and

$$f(\bar{x} + t\bar{h}) > f(\bar{x}).$$

But, the inequality above contradicts the fact that $\bar{x}$ solves problem $(\mathcal{P})$. Hence, Claim 1 is completely proved.

**Claim 2.** $\nabla f(\bar{x}) \cdot h = 0$ for all $h \in \mathbb{R}^n$.

Pick any $h \in \mathbb{R}^n$, $h \neq 0$, and consider the opposite direction $-h \in \mathbb{R}^n$.

By Claim 1, we get $\nabla f(\bar{x}) \cdot h \leq 0$ and $\nabla f(\bar{x}) \cdot (-h) \leq 0$.

Hence, we have that $\nabla f(\bar{x}) \cdot h = 0$.

We conclude that $\nabla f(\bar{x}) \cdot h = 0$ for all $h \in \mathbb{R}^n$, and consequently it must be that $\nabla f(\bar{x}) = 0$. ■

# First order sufficient conditions

If $\bar{x} \in U$ is a **stationary point** of $f$, $\bar{x}$ **is not** necessarily a **solution** of problem $(\mathcal{P})$ (respectively, of problem $(\mathcal{Q})$).

However, under additional conditions one gets the following result.

Let $U$ be an **open and convex** subset of $\mathbb{R}^n$ and let $f$ be a **continuously differentiable** function from $U$ to $\mathbb{R}$.

**Theorem**

- *If $f$ is **concave** in $U$ and $\nabla f(\bar{x}) = 0$, then $\bar{x} \in U$ is a solution of problem $(\mathcal{P})$.*
- *If $f$ is **convex** in $U$ and $\nabla f(\bar{x}) = 0$, then $\bar{x} \in U$ is a solution of problem $(\mathcal{Q})$.*

# Second order necessary conditions for local solutions

We now consider a function $f$ that is $\mathcal{C}^2$ on $U$. We then get information also on the second derivatives of $f$, that is, on the Hessian matrix $D^2 f(\bar{x})$ of $f$ at any local solution $\bar{x}$.

**Theorem**

*If $\bar{x}$ is a **local solution** of problem $(\mathcal{P})$ (respectively, of problem $(\mathcal{Q})$), then $\nabla f(\bar{x}) = 0$ and the Hessian matrix $D^2 f(\bar{x})$ of $f$ at $\bar{x}$ is **negative semi-definite** (respectively, **positive semi-definite**).*

# Second order sufficient conditions for local solutions

Let $f$ be a $\mathcal{C}^2$ function on $U$.

**Theorem**

*If $\bar{x} \in U$ satisfies $\nabla f(\bar{x}) = 0$ and the Hessian matrix $D^2 f(\bar{x})$ is **negative definite** (respectively, **positive definite**), then $\bar{x}$ is a **local solution** of problem $(\mathcal{P})$ (respectively, of problem $(\mathcal{Q})$).*

Differently from second order necessary conditions, **semi-definiteness** is not enough for second order sufficient conditions. Consider for instance the one variable function $f(x) = x^3$. The point 0 is a stationary point of $f$ on the open set $\mathbb{R}$ and $D^2 f(0) = 0$, but 0 is not a (local) solution of problem $(\mathcal{P})$ (respectively, of problem $(\mathcal{Q})$).

# Optimization problems with equality constraints

Let $U$ be an **open** subset of $\mathbb{R}^n$. The functions $f$ and $g_1, \ldots, g_i, \ldots, g_p$ are defined on $U$. We consider the following optimization problems with **equality constraints**.

$$(\mathcal{P}) \begin{cases} \max\limits_{x \in U} f(x) \\ g_i(x) = 0,\ i = 1, \ldots, p \end{cases} \qquad (\mathcal{Q}) \begin{cases} \min\limits_{x \in U} f(x) \\ g_i(x) = 0,\ i = 1, \ldots, p \end{cases}$$

## Definition

Assume that $g_1, \ldots, g_i, \ldots, g_p$ are $\mathcal{C}^1$ on $U$. Let $\bar{x} \in U$ be a point such that

$$g_i(\bar{x}) = 0, \text{ for all } i = 1, \ldots, p.$$

The **constraint qualification condition** is satisfied at $\bar{x}$ if all the gradient vectors $\nabla g_1(\bar{x}), \ldots, \nabla g_i(\bar{x}), \ldots, \nabla g_p(\bar{x})$ are **linearly independent**.

# First order necessary conditions

## Theorem

*Assume that the functions $f$ and $g_1, \ldots, g_i, \ldots, g_p$ are $\mathcal{C}^1$ on $U$.*

*Let $\bar{x} \in U$ be a* ***solution*** *of problem $(\mathcal{P})$ (resp., problem $(\mathcal{Q})$) that satisfies the* ***constraint qualification condition****.*

*Then, there exists a vector of Lagrange multipliers $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_i, \ldots, \bar{\lambda}_p) \in \mathbb{R}^p$ such that :*

$$\nabla f(\bar{x}) - \sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0$$

# A counterexample

## Remark

*The previous result does not hold true if the constraint qualification condition is not satisfied. Indeed, consider the following minimization problem :*

$$\begin{cases} \min\limits_{(x,y)\in\mathbb{R}^2} f(x,y) = x + y \\ g_1(x,y) = (x-1)^2 + y^2 - 1 = 0 \\ g_2(x,y) = (x+1)^2 + y^2 - 1 = 0 \end{cases}$$

$\{(x,y) \in \mathbb{R}^2 \mid g_1(x,y) = g_2(x,y) = 0\} = \{(0,0)\} =$ *Set of solutions.*

$\nabla f(0,0) = (1,1)$, $\nabla g_1(0,0) = (-2,0)$, $\nabla g_2(0,0) = (2,0)$.

$\nabla g_1(0,0)$ *and* $\nabla g_2(0,0)$ *are* ***not linearly independent****, because* $\nabla g_2(0,0) = -\nabla g_1(0,0)$.

$\nexists(\lambda_1,\lambda_2) \in \mathbb{R}^2$ *such that* $\nabla f(0,0) = \lambda_1 \nabla g_1(0,0) + \lambda_2 \nabla g_2(0,0)$.

# Lagrangian function

## Definition

The **Lagrangian function** $\mathcal{L}$ associated with problem $(\mathcal{P})$ (resp., problem $(\mathcal{Q})$) is the function from $U \times \mathbb{R}^p$ to $\mathbb{R}$ defined by :

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^{p} \lambda_i g_i(x),$$

where $\lambda = (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_p) \in \mathbb{R}^p$.

Notice that for every $(\bar{x}, \bar{\lambda}) \in U \times \mathbb{R}^p$ :

$$\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \nabla f(\bar{x}) - \sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x}), \text{ and}$$

$$\forall i = 1, \ldots, p, \ \frac{\partial \mathcal{L}}{\partial \lambda_i}(\bar{x}, \bar{\lambda}) = -g_i(\bar{x}).$$

# First order sufficient conditions

Let $U$ be an **open and convex** subset of $\mathbb{R}^n$.

## Theorem

*Assume that the functions $f$ and $g_1, \ldots, g_i, \ldots, g_p$ are $\mathcal{C}^1$ on $U$. Let $\bar{x} \in U$ be a point such that $g_i(\bar{x}) = 0$ for all $i = 1, \ldots, p$.*

*If there exists a vector of Lagrange multipliers $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_i, \ldots, \bar{\lambda}_p) \in \mathbb{R}^p$ such that*

$$\nabla f(\bar{x}) - \sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0,$$

*and the Lagrangian function $\mathcal{L}$ is **concave** (resp., **convex**) in $x$, then $\bar{x}$ is a **solution** of problem $(\mathcal{P})$ (resp., problem $(\mathcal{Q})$).*

# Second order necessary conditions for local solutions

Assume that $f$ and $g_1, \ldots, g_i, \ldots, g_p$ are $\mathcal{C}^2$ on $U$. Consider $\bar{x} \in U$ and the following set :

$$Z(\bar{x}) = \{z \in \mathbb{R}^n \mid \nabla g_i(\bar{x}) \cdot z = 0, \forall i = 1, \ldots, p\}.$$

Denote by $D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda})$ the **partial Hessian matrix of the Lagrangian function** $\mathcal{L}(\cdot, \bar{\lambda})$ with respect to $x$ at $\bar{x}$.

## Theorem

*Let $\bar{x}$ be a **local solution** of problem $(\mathcal{P})$ (resp., problem $(\mathcal{Q})$) that satisfies the **constraint qualification condition**, and $\bar{\lambda} \in \mathbb{R}^p$ such that $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = 0$.*

*Then, $D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda})$ is **negative semi-definite** (resp., **positive semi-definite**) on $Z(\bar{x})$, i.e.,*

$$z^T D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda}) z \leq 0 \ (\textit{resp.} \geq 0), \ \textit{for all } z \in Z(\bar{x}).$$

# Second order sufficient conditions for local solutions

Assume that $f$ and $g_1, \ldots, g_i, \ldots, g_p$ are $\mathcal{C}^2$ on $U$.

**Theorem**

*Let $\bar{x} \in U$ such that $g_i(\bar{x}) = 0$ for all $i = 1, ..., p$, and $\nabla f(\bar{x}) - \sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0$, for some $\bar{\lambda} \in \mathbb{R}^p$. If*

$$z^T D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda}) z < 0 \ (\textit{resp.} > 0), \ \textit{for all } z \in Z(\bar{x}) \setminus \{0\},$$

*that is, if the partial Hessian matrix of the Lagrangian function $\mathcal{L}(\cdot, \bar{\lambda})$ at $\bar{x}$, i.e., $D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda})$ is **negative definite** (resp., **positive definite**) definite on the set $Z(\bar{x}) \setminus \{0\}$, then $\bar{x}$ is a **local solution** of problem $(\mathcal{P})$ (resp., problem $(\mathcal{Q})$).*

# Conditions on the border Hessian determinants

We assume that the **constraint qualification condition** is satisfied at $\bar{x} \in U$.

Consider the mapping $g = (g_1, \ldots, g_i, \ldots, g_p)$ defined by :

$$g : x \in U \subseteq \mathbb{R}^n \longrightarrow g(x) = (g_1(x), \ldots, g_i(x), \ldots, g_p(x)) \in \mathbb{R}^p$$

We rank the components of $x$ in such a way that the first $p$ columns of the Jacobian matrix $Dg(\bar{x})$ are linearly independent.

This is possible because the Jacobian matrix $Dg(\bar{x})$ has rank $p$, since its rows are the gradients of the constraint functions.

For every $r = p+1, \ldots, n$, define the following **border Hessian determinant** :

$$B_r(\bar{x}) = \begin{vmatrix} 0 & \cdots & 0 & \frac{\partial g_1(\bar{x})}{\partial x_1} & \cdots & \frac{\partial g_1(\bar{x})}{\partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{\partial g_p(\bar{x})}{\partial x_1} & \cdots & \frac{\partial g_p(\bar{x})}{\partial x_r} \\ \frac{\partial g_1(\bar{x})}{\partial x_1} & \cdots & \frac{\partial g_p(\bar{x})}{\partial x_1} & \frac{\partial^2 \mathcal{L}(\bar{x})}{\partial x_1^2} & \cdots & \frac{\partial^2 \mathcal{L}(\bar{x})}{\partial x_1 \partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_1(\bar{x})}{\partial x_r} & \cdots & \frac{\partial g_p(\bar{x})}{\partial x_r} & \frac{\partial^2 \mathcal{L}(\bar{x})}{\partial x_r \partial x_1} & \cdots & \frac{\partial^2 \mathcal{L}(\bar{x})}{\partial x_r^2} \end{vmatrix}$$

## Proposition

- *If for all* $r = p+1, \ldots, n$, $(-1)^r B_r(\bar{x}) > 0$, *then the partial Hessian matrix of the Lagrangian function* $\mathcal{L}(\cdot, \bar{\lambda})$ *at* $\bar{x}$, *i.e.,* $D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda})$ *is* ***negative definite*** *on* $Z(\bar{x}) \setminus \{0\}$.

- *If for all* $r = p+1, \ldots, n$, $(-1)^p B_r(\bar{x}) > 0$, *then the partial Hessian matrix of the Lagrangian function* $\mathcal{L}(\cdot, \bar{\lambda})$ *at* $\bar{x}$, *i.e.,* $D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda})$ *is* ***positive definite*** *on* $Z(\bar{x}) \setminus \{0\}$.

# Interpretation of the Lagrange multipliers

Consider $b = (b_1, ..., b_i, ..., b_p) \in \mathbb{R}^p$ and the "perturbed" problem $(\mathcal{P}_b)$ of problem $(\mathcal{P})$ :

$$(\mathcal{P}_b) \begin{cases} \max\limits_{x \in U} f(x) \\ g_i(x) = b_i,\ i = 1, \ldots, p \end{cases}$$

Assume that the value $v(b)$ of problem $(\mathcal{P}_b)$ is a **well-defined** function around $0 \in \mathbb{R}^p$. That is, there exists an open ball $B \subseteq \mathbb{R}^p$ of center 0 such that for all $b \in B$, problem $(\mathcal{P}_b)$ has at least a solution in $U$.

For all $b \in B$, the **value function** $v(b)$ is then defined by :

$$v(b) = \max\{f(x) \colon x \in U \text{ and } g_i(x) = b_i, \forall i = 1, \ldots, p\}$$

Also assume that the value function $v$ is **differentiable** on $B$.

Let $\bar{x}$ be a solution of problem $(\mathcal{P}_0)$. Consider the mapping $g = (g_1, \ldots, g_i, \ldots, g_p)$ from $U$ to $\mathbb{R}^p$ and notice that $g(\bar{x}) = 0$.

For all $x$ in a open neighborhood of $\bar{x}$, define $V$ as follows :

$$V(x) := f(x) - v(g(x)) \leq 0 \text{ and } V(\bar{x}) = 0.$$

Then $\nabla V(\bar{x}) = 0$, because $\bar{x}$ maximizes the function $V$ in an open set.

From the **chain rule** for differentiable mappings, one gets :

$$0 = \nabla V(\bar{x}) = \nabla f(\bar{x}) - [Dg(\bar{x})]^T \nabla_b v(0).$$

Equivalently,

$$\nabla f(\bar{x}) = \sum_{i=1}^{p} \frac{\partial v}{\partial b_i}(0) \nabla g_i(\bar{x}).$$

Let $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_i, \ldots, \bar{\lambda}_p) \in \mathbb{R}^p$ the Lagrange multipliers associated with the solution $\bar{x}$ of problem $(\mathcal{P}_0)$. That is,

$$\nabla f(\bar{x}) = \sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x})$$

Hence, for each $i = 1, \ldots, p$, the Lagrange multiplier $\bar{\lambda}_i$ is equal to the partial derivative $\dfrac{\partial v}{\partial b_i}(0)$ of the value function $v$ at $b = 0$.