The variance as a risk measure

A brief introduction to Markowitz's portfolio theory.

Noufel Frikha

Université Paris 1 Panthéon-Sorbonne

November 2025



Contents

1. Introduction
2. Mathematical framework
3. A brief economical derivation of Markowitz's problem
4. Markowitz with no-risk free asset
5. Markowitz with a risk-free asset
6. Practical implementation and limitations of Markowitz' approach

Introduction

A first example: two risky assets

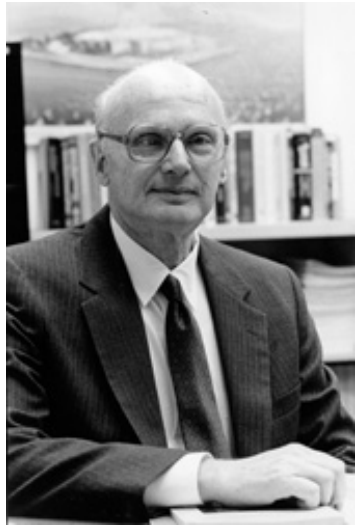
Consider simple returns over the same horizon:

	Asset 1	Asset 2
Expected return	5%	8%
Standard deviation	10%	20%

With weights w and $1-w$, where $0 \leq w \leq 1$,

$$R_p = wR_1 + (1-w)R_2, \qquad \mathbb{E}[R_p] = 0.05w + 0.08(1-w).$$

Can asset 2 improve a portfolio even though it is more volatile?



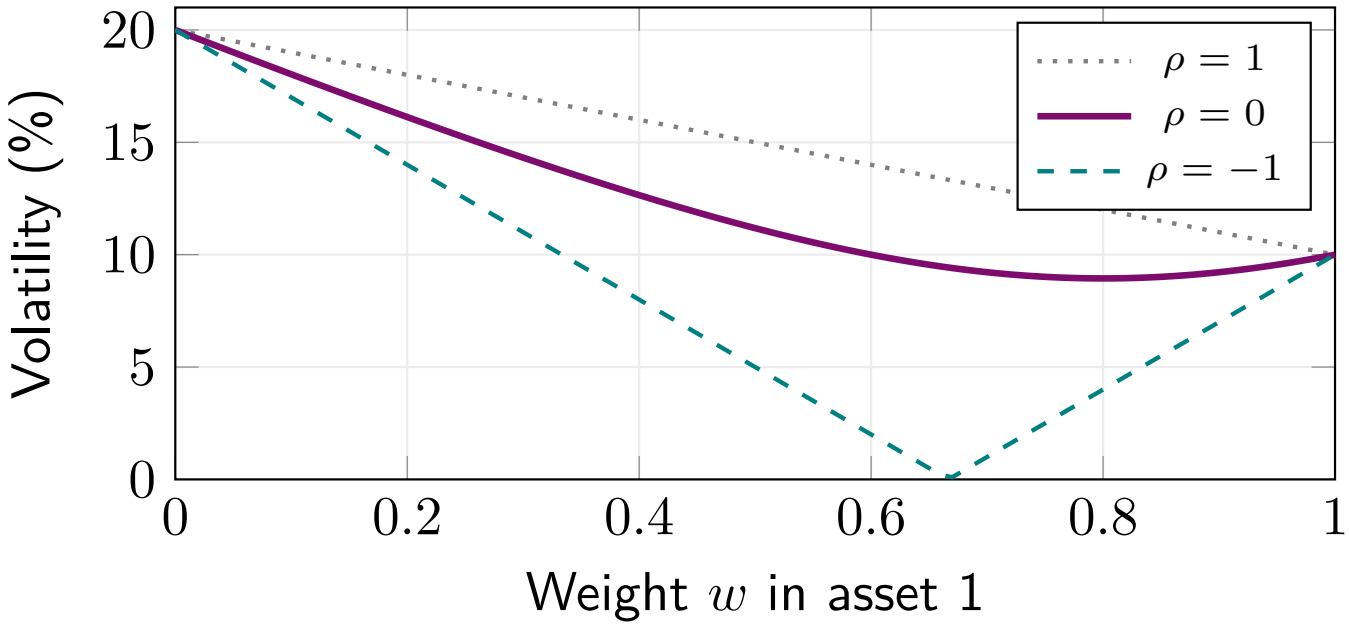
Harry Markowitz

Portrait: Henry Foster, undated. University of Chicago.

Diversification depends on correlation

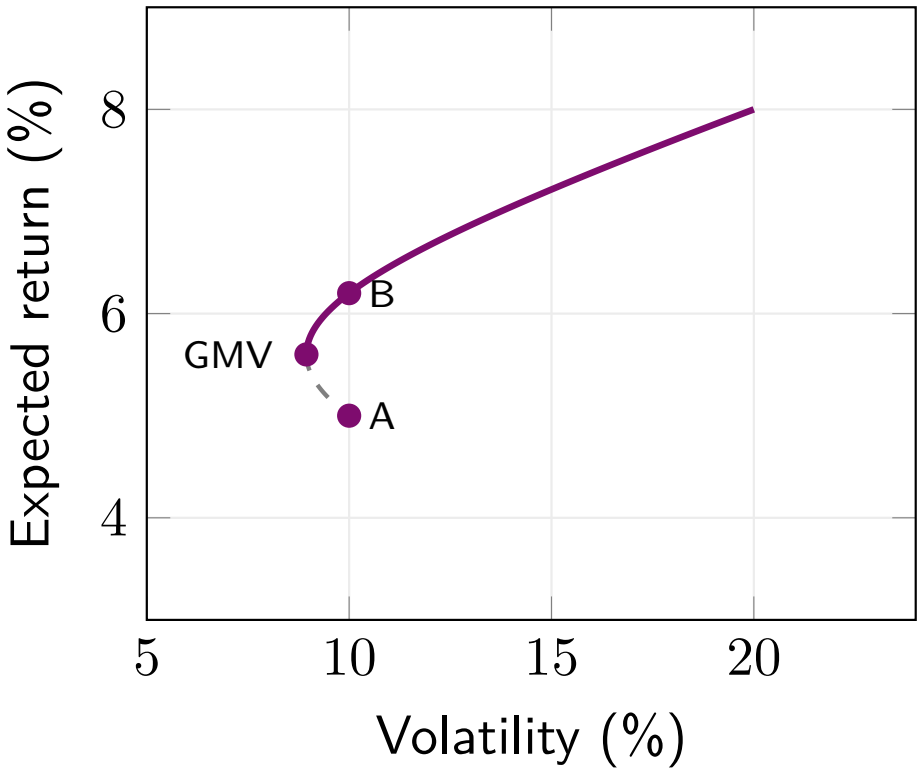
For $\rho = \mathrm{Corr}(R_1, R_2)$,

$$\sigma_p^2 = 0.01w^2 + 0.04(1-w)^2 + 0.04\rho w(1-w).$$



With $\rho = 0$, an 80/20 allocation has volatility $\sqrt{0.008} \simeq 8.94\%$, below either asset's volatility. Perfect negative correlation is a limiting, singular case.

Which portfolios can we rule out?



A feasible portfolio need not be efficient.

Same two assets, $\rho = 0$ and $w \in [0, 1]$.

- A and B have the same volatility; B has a higher expected return.
- The GMV portfolio has the lowest variance.
- The solid branch is efficient; the dashed branch is dominated.

Introduction

• **Description of the market:** We consider a financial market with two dates $t = 0$ and $t = 1$. The financial market is composed of:

- 1 non-risky asset. Its price at time t is denoted by S_t^0 (Bond price).
- d risky assets. Their prices at time t are denoted by $(S_t^1, \cdots, S_t^d)$ (Stock prices).

You invest your initial wealth V_0 at time 0 and hold position up to time 1. Here, $(S_1^0, \cdots, S_1^d)$ are random variables, leaving on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, which satisfy some assumptions that will be specified later on.

• **Aim:** Starting from an initial wealth V_0, we want to invest in the non-risky and risky assets in such a way that optimizes our terminal wealth V_1.

• **Questions:**

- How would you invest? Maximize expected return of the portfolio but limit/control risk.
- How to measure risk? As originally proposed by H. Markowitz, we can measure the risk of a strategy using its variance/standard deviation.

Mathematical framework

- We will not directly work with prices:
  - Returns:

$$\text{(interest rate)} \quad \frac{S_1^0 - S_0^0}{S_0^0} = r, \quad \text{(return of asset } i\text{)} \quad R^i = \frac{S_1^i - S_0^i}{S_0^i}.$$

  - Investment strategy/portfolio is determined by
    - an initial wealth V_0
    - a vector of weights $(w_0, \cdots, w_d)$ where w_i stands for the proportion of initial wealth invested in asset i:

$$\text{(Budget constraint)} \quad \sum_{i=0}^{d} w_i = 1.$$

Note that $w_i \leq 0$ means short position while $w_i \geq 0$ means long position in asset i. The proportion invested in the non-risky asset is entirely determined by $w = (w_1, \cdots, w_d)$ since

$$w_0 = 1 - \mathbf{1}^t w, \quad \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^d.$$

Mathematical framework

Mathematical framework

- The terminal wealth is given by

$$V_1 = \sum_{i=0}^{d} \delta^i S_1^i = V_0 \sum_{i=0}^{d} w_i \frac{S_1^i}{S_0^i} \quad \text{using that } \delta^i = \frac{w_i V_0}{S_0^i}$$

so that

$$\frac{V_1}{V_0} = \frac{S_1^0}{S_0^0} + \sum_{i=1}^{d} w_i \Big(\frac{S_1^i}{S_0^i} - \frac{S_1^0}{S_0^0} \Big) = 1 + r + \sum_{i=1}^{d} w_i (R^i - r)$$

which eventually gives

$$\frac{V_1 - V_0}{V_0} - r = \sum_{i=1}^{d} w_i (R^i - r).$$

$\rightsquigarrow$ Markowitz problem: Given an initial wealth V_0, find a strategy w such that

$$\text{(Maximize expected excess return)} \quad \max_{w \in \mathbb{R}^d} \mathbb{E}\Big[\frac{V_1 - V_0}{V_0} - r \Big]$$

while controlling the risk measured by the variance $var(V_1/V_0) := \sigma^2(w)$.

◦ We solve the following optimization problem:

$$\max_{w\in\mathbb{R}^d}\left\{R(w):=\mathbb{E}\Big[\frac{V_1-V_0}{V_0}-r\Big]\right\} \quad \text{such that}$$

such that

$$\begin{cases}\sigma^2(w)=var(\frac{V_1-V_0}{V_0}-r)\leq\sigma_\star^2,\\ w\in B,\end{cases}$$

where $\sigma_\star^2>0$ is a given risk level that we do not want to exceed and B is a set of constraints.

◦ Assumption on $R=(R^1,\cdots,R^d)$: First two moments exist

- $\mu:=\mathbb{E}[R-r\mathbf{1}]$ is the expected excess return of the risky assets
- $\Sigma:=Cov((R-r\mathbf{1})^i,(R-r\mathbf{1})^j)_{1\leq i,j\leq d}\in\mathbb{S}_+^d$, where $\mathbb{S}_+^d$ is the set of $d\times d$ symmetric semi-definite non-negative matrix.

- We now compute the (excess) mean return and variance of the portfolio:

$$\mathbb{E}\Big[\frac{V_1 - V_0}{V_0} - r\Big] = \sum_{i=1}^{d} w_i \mathbb{E}[R^i - r] = w^t \mu$$

and

$$\begin{aligned}
\sigma^2(w) &:= var(\sum_{i=1}^{d} w_i (R^i - r)) \\
&= var(w^t (R - r\mathbf{1})) \\
&= \mathbb{E}\Big[(w^t (R - r\mathbf{1} - \mathbb{E}[R - r\mathbf{1}]))(w^t (R - r\mathbf{1} - \mathbb{E}[R - r\mathbf{1}]))\Big] \\
&= \mathbb{E}\Big[(w^t (R - r\mathbf{1} - \mathbb{E}[R - r\mathbf{1}]))((R - r\mathbf{1} - \mathbb{E}[R - r\mathbf{1}])^t w)\Big] \\
&= w^t \Sigma w.
\end{aligned}$$

Hence, the Markowitz optimization problem writes

$$\max_{w \in \mathbb{R}^d} w^t \mu, \quad s.t. \quad w^t \Sigma w \leq \sigma_\star^2, \quad \text{and} \quad w \in B.$$

What type of constraints on the weights

(i) Unconstrained: $w \in \mathbb{R}^d$

What type of constraints on the weights

(i) Unconstrained: $w \in \mathbb{R}^d$

(ii) No-investment in the non-risky asset: $w_0 = 0 \Leftrightarrow \sum_{i=1}^{d} w_i = 1 \Leftrightarrow w^t \mathbf{1} = 1.$

What type of constraints on the weights

(i) Unconstrained: $w \in \mathbb{R}^d$

(ii) No-investment in the non-risky asset: $w_0 = 0 \Leftrightarrow \sum_{i=1}^{d} w_i = 1 \Leftrightarrow w^t \mathbf{1} = 1$.

(iii) No-shorting constraint: $w_i \geq 0$ for all $i = 1, \cdots, d$.

What type of constraints on the weights

(i) Unconstrained: $w \in \mathbb{R}^d$

(ii) No-investment in the non-risky asset: $w_0 = 0 \Leftrightarrow \sum_{i=1}^d w_i = 1 \Leftrightarrow w^t \mathbf{1} = 1$.

(iii) No-shorting constraint: $w_i \geq 0$ for all $i = 1, \cdots, d$.

(iv) ℓ^p-constraints: $\|w\|_{\ell^p} := (\sum_{i=1}^d |w_i|^p)^{1/p} \leq \alpha$ for e.g. $p = 1$ or $p = 2$.

What type of constraints on the weights

(i) Unconstrained: $w \in \mathbb{R}^d$

(ii) No-investment in the non-risky asset: $w_0 = 0 \Leftrightarrow \sum_{i=1}^d w_i = 1 \Leftrightarrow w^t \mathbf{1} = 1$.

(iii) No-shorting constraint: $w_i \geq 0$ for all $i = 1, \cdots, d$.

(iv) ℓ^p-constraints: $\|w\|_{\ell^p} := (\sum_{i=1}^d |w_i|^p)^{1/p} \leq \alpha$ for e.g. $p = 1$ or $p = 2$.

We will solve the Markowitz optimization problem under (i) or (ii).

A brief economical derivation of Markowitz's problem

Economical derivation of Markowitz's problem

◦ Gaussian returns and exponential utility provide one justification for this criterion.

◦ Assume that $R \sim \mathcal{N}(m, \Sigma)$, with $\mu = m - r\mathbf{1}$, and that the agent preference ordering over different investment strategies is based on the utility function

$$U(x) = 1 - \exp(-\lambda x), \text{ for some parameter } \lambda > 0.$$

Note that $\lambda = -U''/U'$ is called the risk aversion parameter.

◦ The optimization problem of the agent will be to maximize his satisfaction in the future, namely

$$\max_{w \in \mathbb{R}^d} \mathbb{E}\Big[U\Big(\frac{V_1}{V_0}\Big)\Big], \qquad \text{where } \frac{V_1}{V_0} = 1 + r + w^t(R - r\mathbf{1}) \sim \mathcal{N}(1 + r + w^t\mu, w^t\Sigma w)$$

and one has

$$\mathbb{E}\Big[U\Big(\frac{V_1}{V_0}\Big)\Big] = 1 - \mathbb{E}\Big[e^{-\lambda \frac{V_1}{V_0}}\Big] = 1 - e^{-\lambda(1+r) - \lambda w^t\mu + \frac{\lambda^2}{2} w^t\Sigma w}.$$

- The previous optimization problem is equivalent to

$$\max_{w\in\mathbb{R}^d}\left\{\mathcal{L}(w,\lambda) := w^t\mu - \frac{\lambda}{2}w^t\Sigma w\right\}$$

which is equivalent to the Lagrangian formulation of Markowitz's optimization problem.

- The Lagrange multiplier λ corresponds to the risk aversion parameter of the agent and is directly linked to the level of maximal risk $\sigma_\star^2$ the agent is willing to take.

Markowitz with no-risk free asset

◦ We fix the risk aversion parameter $\lambda > 0$. Corresponding to the constrained optimization problem

$$\max_{w:\mathbf{1}^t w=1} w^t\mu - \frac{\lambda}{2}(w^t\Sigma w - \sigma_\star^2),$$

we associate the Lagrangian

$$\mathcal{L}(w,\beta) := w^t\mu - \frac{\lambda}{2}(w^t\Sigma w - \sigma_\star^2) - \beta(w^t\mathbf{1} - 1).$$

◦ Assumption: $\Sigma \succ 0$: every nonzero linear combination of centred risky returns has strictly positive variance or, in other words, no nonzero combination of risky-asset returns can eliminate all uncertainty.

◦ Karush Kuhn Tucker optimality conditions:

$$\begin{cases} \nabla_w \mathcal{L}(w,\beta) = 0 \\ \partial_\beta \mathcal{L}(w,\beta) = 0 \end{cases} \Leftrightarrow \begin{cases} \mu - \lambda\Sigma w - \beta\mathbf{1} = 0 \\ w^t\mathbf{1} = 1 \end{cases} \Leftrightarrow \begin{cases} w = \frac{1}{\lambda}\Sigma^{-1}(\mu - \beta\mathbf{1}) \\ w^t\mathbf{1} = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} w = \frac{1}{\lambda}\Sigma^{-1}(\mu - \beta\mathbf{1}) \\ \beta = \frac{\mathbf{1}^t\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} - \frac{\lambda}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} \end{cases} \Leftrightarrow \left\{ w_\star = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} + \frac{1}{\lambda}\left(\Sigma^{-1}\mu - \frac{\mathbf{1}^t\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}\right) \right.$$

- Assume $b := \mathbf{1}^t \Sigma^{-1} \mu > 0$ for the tangency-portfolio representation below.
- Examples:
  - Maximal risk aversion: $\lambda \uparrow \infty$ then

$$w_\star = w_{\mathrm{GMV}} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^t \Sigma^{-1}\mathbf{1}}$$

    which formally corresponds to the solution of the unconstrained optimization problem $\min_w w^t \Sigma w$ s.t. $\mathbf{1}^t w = 1$.
  - Tangency portfolio: For $\lambda = \mathbf{1}^t \Sigma^{-1} \mu$, we obtain

$$w_\star = w_T = \frac{\Sigma^{-1}\mu}{\mathbf{1}^t \Sigma^{-1}\mu}$$

    which corresponds to the portfolio fully invested in risky assets (in the presence of a non-risky asset).
- Going back to the original $w_\star$, we observe that

$$w_\star = (1 - \frac{\mathbf{1}^t \Sigma^{-1}\mu}{\lambda}) \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^t \Sigma^{-1}\mathbf{1}} + \frac{\mathbf{1}^t \Sigma^{-1}\mu}{\lambda} \frac{\Sigma^{-1}\mu}{\mathbf{1}^t \Sigma^{-1}\mu} = (1-\alpha) w_{\mathrm{GMV}} + \alpha w_T$$

with $\alpha = \frac{\mathbf{1}^t \Sigma^{-1}\mu}{\lambda}$. This result is known as the two fund theorem.

Interpreting the minimum-variance portfolio

For two uncorrelated assets and weights summing to one,

$$\min_{w\in\mathbb{R}}\left\{w^2\sigma_1^2+(1-w)^2\sigma_2^2\right\} \quad\Longrightarrow\quad w_1^{\mathrm{GMV}}=\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}, \qquad w_2^{\mathrm{GMV}}=\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}.$$

For $\sigma_1=10\%$ and $\sigma_2=20\%$,

$$w^{\mathrm{GMV}}=(0.8,0.2), \qquad \sigma_{\mathrm{GMV}}\simeq 8.94\%, \qquad \mathbb{E}[R_{\mathrm{GMV}}]=5.6\%.$$

- The more volatile asset still receives a positive weight.
- Minimum variance does not mean investing only in the least volatile asset.
- Expected returns do not enter the GMV weights.

Individual volatility and portfolio contribution are different concepts.

Two fund theorem with only risky assets

Theorem

Assume $\Sigma \succ 0$ and $b = \mathbf{1}^t \Sigma^{-1} \mu > 0$. For $\lambda > 0$, the solution $w_\star$ to the Markowitz optimization problem with only risky-assets can be represented as a linear combination of only two portfolios: the global minimum-variance portfolio and the tangency portfolio.

$$w_\star = (1-\alpha) w_{\mathrm{GMV}} + \alpha w_T, \qquad \text{with} \qquad \alpha = \frac{\mathbf{1}^t \Sigma^{-1} \mu}{\lambda}.$$

◦ Remarks:

- As $\lambda \uparrow \infty$, $\alpha \to 0$, the portfolio becomes the GMV portfolio.
- If $\lambda = \mathbf{1}^t \Sigma^{-1} \mu$ then $\alpha = 1$, the portfolio becomes the tangency portfolio.

Decomposing the optimal portfolio

Assume $\Sigma \succ 0$ and $\lambda > 0$. Introduce the scalars

$$a = \mathbf{1}^t \Sigma^{-1} \mathbf{1}, \qquad b = \mathbf{1}^t \Sigma^{-1} \mu, \qquad c = \mu^t \Sigma^{-1} \mu.$$

The optimal fully invested portfolio can be written as

$$\boxed{w_\lambda = w_{\mathrm{GMV}} + \frac{1}{\lambda} z},$$

where

$$w_{\mathrm{GMV}} = \frac{\Sigma^{-1}\mathbf{1}}{a}, \qquad z = \Sigma^{-1}\mu - \frac{b}{a}\Sigma^{-1}\mathbf{1}.$$

Since $\mathbf{1}^t z = b - (b/a)a = 0$, the position z has zero net investment. Starting from GMV, we reallocate wealth without changing the total budget.

Why the covariance term vanishes

Since $\Sigma w_{\mathrm{GMV}} = \mathbf{1}/a$ and $\mathbf{1}^t z = 0$,

$$\boxed{w_{\mathrm{GMV}}^t \Sigma z = \frac{1}{a}\mathbf{1}^t z = 0}.$$

The GMV return and the return of the position z are uncorrelated. This does not imply independence.

Set $\Delta = c - b^2/a \geq 0$. Then

$$\mu^t z = c - \frac{b^2}{a} = \Delta,$$

$$z^t \Sigma z = z^t \left(\mu - \frac{b}{a}\mathbf{1}\right) = z^t \mu = \Delta.$$

The zero-budget identity eliminates the cross term.

Mean and variance of the optimal portfolio

Recall $w_\lambda = w_{\mathrm{GMV}} + z/\lambda$ and $\Delta = c - b^2/a$.

Expected excess return:

$$R_\star = \mu^t w_\lambda = \underbrace{\frac{b}{a}}_{R_{\mathrm{GMV}}} + \frac{\Delta}{\lambda}.$$

Variance: the cross term is zero, hence

$$\sigma_\star^2 = w_{\mathrm{GMV}}^t \Sigma w_{\mathrm{GMV}} + \frac{2}{\lambda} \underbrace{w_{\mathrm{GMV}}^t \Sigma z}_{0} + \frac{1}{\lambda^2} z^t \Sigma z$$

$$= \underbrace{\frac{1}{a}}_{\sigma^2_{\mathrm{GMV}}} + \frac{\Delta}{\lambda^2}.$$

The mean changes linearly with $1/\lambda$; additional variance changes quadratically.

From optimal moments to the efficient frontier

For $\Delta > 0$, eliminate λ from

$$R_\star - R_{\mathrm{GMV}} = \frac{\Delta}{\lambda}, \qquad \sigma_\star^2 - \sigma_{\mathrm{GMV}}^2 = \frac{\Delta}{\lambda^2}.$$

Since $\lambda > 0$,

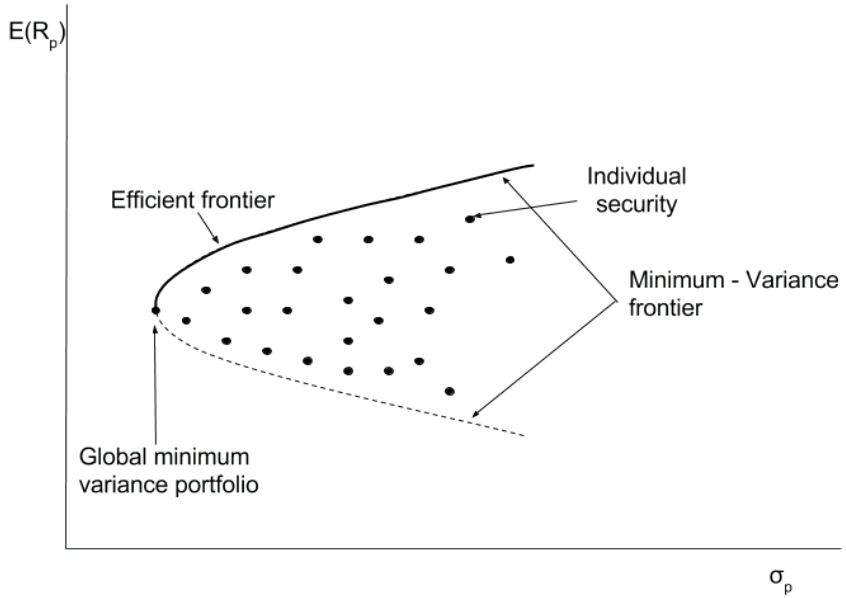
$$\boxed{R_\star = R_{\mathrm{GMV}} + \sqrt{\Delta}\sqrt{\sigma_\star^2 - \sigma_{\mathrm{GMV}}^2}}.$$

- $\Delta \geq 0$ follows from Cauchy–Schwarz in the Σ^{-1} inner product.
- $\Delta = 0$ means μ is proportional to $\mathbf{1}$: all fully invested portfolios have the same expected return, and GMV is optimal.

The geometry of the frontier

For $\Delta > 0$, the minimum-variance frontier satisfies

$$\frac{\sigma^2}{\sigma^2_{\mathrm{GMV}}} - \frac{(R - R_{\mathrm{GMV}})^2}{\Delta \sigma^2_{\mathrm{GMV}}} = 1.$$



In the (σ, R) plane, the feasible frontier is the right branch of a hyperbola. Only its upper part, $R \geq R_{\mathrm{GMV}}$, is efficient. The illustration uses total return; our formulas use excess return.

Comments on the efficient frontier

◦ Concave relation between $R_\star$ and $\sigma_\star$ reflects the idea of expected utility concavely increasing with level of risk.

◦ For a given level of risk, the efficient portfolio has the highest expected return $R_\star$ in all the investment universe (i.e. exclusively invested in stocks). Said differently, for a given level of return R, the efficient portfolio has the lowest level of risk.

Markowitz with a risk-free asset

Markowitz with a risk-free asset

We now allow investment in a risk-free asset. Fix $\lambda > 0$ and assume $b = \mathbf{1}^t \Sigma^{-1} \mu > 0$ for the tangency representation. The optimization problem writes

$$\max_{w \in \mathbb{R}^d} \left\{ \mathcal{L}(w) := w^t \mu - \frac{\lambda}{2} w^t \Sigma w \right\}$$

Recall from the budget constraint that $w_0 = 1 - w^t \mathbf{1}$ is the proportion invested in the risk-free asset.

- Assumption: $\Sigma \succ 0$ (positive definite).

The KKT optimality condition gives

$$\nabla_w \mathcal{L}(w^\star) = 0 \Leftrightarrow \Sigma w^\star = \frac{1}{\lambda} \mu \Leftrightarrow w^\star = \frac{1}{\lambda} \Sigma^{-1} \mu.$$

- The portfolio is exclusively invested in risky assets if and only if

$$1 = \mathbf{1}^t w^\star \Leftrightarrow \lambda = \mathbf{1}^t \Sigma^{-1} \mu \Rightarrow w_T = \frac{\Sigma^{-1} \mu}{\mathbf{1}^t \Sigma^{-1} \mu} \quad \text{(Tangency portfolio)}$$

Two fund theorem with a risk-free asset

Theorem

Assume $b = \mathbf{1}^t \Sigma^{-1} \mu > 0$. *The unique solution* $w^\star$ *to the Markowitz optimization problem, where investment in the non-risky asset is allowed, can be represented as a combination of two portfolios: the tangency portfolio and the non-risky asset*

$$w^\star = \frac{\mathbf{1}^t \Sigma^{-1} \mu}{\lambda} \frac{\Sigma^{-1} \mu}{\mathbf{1}^t \Sigma^{-1} \mu} = \alpha_T \frac{\Sigma^{-1} \mu}{\mathbf{1}^t \Sigma^{-1} \mu} = \alpha_T w_T, \quad \text{with} \quad \alpha_T = \frac{\mathbf{1}^t \Sigma^{-1} \mu}{\lambda}$$

and

$$w_0 = 1 - \mathbf{1}^t w^\star = 1 - \alpha_T.$$

- Remarks: Particular cases are:
  - As $\lambda \uparrow \infty$, $\alpha_T \to 0$ (fully invested in non-risky asset)
  - For $\lambda = \mathbf{1}^t \Sigma^{-1} \mu$, $\alpha_T = 1$ (fully invested in the tangency portfolio).

Capital Market Line

The mean and variance of optimal portfolios are

$$R^\star = \mu^t w^\star = \frac{1}{\lambda}\mu^t\Sigma^{-1}\mu, \quad \text{and} \quad (\sigma^\star)^2 = \frac{1}{\lambda^2}\mu^t\Sigma^{-1}\mu.$$

Hence, we obtain

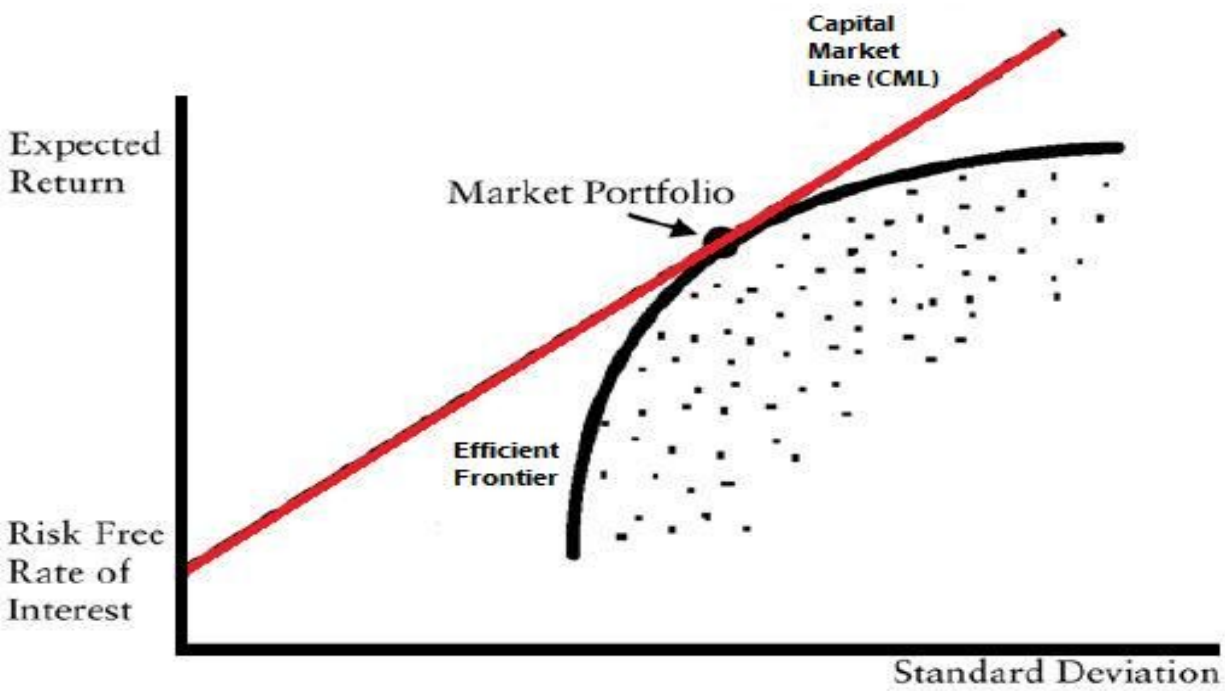
$$R^\star = \sqrt{\mu^t\Sigma^{-1}\mu} \times \sigma^\star.$$

This equation represents a line in the (σ, R) plane, known as the Capital Market Line, which passes through the following two points/portfolios:

- $\sigma^\star = 0$ and $R^\star = 0$. This corresponds to the return of the risk-free asset.
- $\sigma^\star = \sigma_T$ and $R^\star = R_T$, which corresponds to the Tangency portfolio that lies on the efficient frontier.

From our previous analysis, the CML touches the efficient frontier in only one point, the tangency portfolio.

Capital Market Line



The figure labels the tangency portfolio as "Market Portfolio". Identifying it with the market portfolio requires additional equilibrium assumptions (CAPM).

Sharpe ratio

Based on the two moments, one can define a performance measure known as the Sharpe ratio of the portfolio.

$$\mathcal{S}_p = \frac{R_p}{\sigma_p}.$$

◦ Interpretation: it represents the excess return above the risk-free rate per unit of risk, where the risk is measured by the standard deviation. It represents the additional amount of return an investor receives per unit of increase in risk. When comparing two assets, the one with a higher Sharpe ratio appears to provide a better return for the same risk, which is usually attractive for investors.

◦ Sharpe ratio of portfolios on the CML: Recalling that $R^\star = \sqrt{\mu^t \Sigma^{-1} \mu} \times \sigma^\star$, one has

$$\mathcal{S}_{CML} = \sqrt{\mu^t \Sigma^{-1} \mu}.$$

A Sharpe-ratio bound: Cauchy–Schwarz

Assume $\Sigma \succ 0$. For any nonzero risky position w,

$$\mathcal{S}(w) = \frac{w^t \mu}{\sqrt{w^t \Sigma w}}.$$

Factor the numerator as

$$w^t \mu = (\Sigma^{1/2} w)^t (\Sigma^{-1/2} \mu).$$

Cauchy–Schwarz gives

$$|w^t \mu| \leq \underbrace{\|\Sigma^{1/2} w\|}_{\sqrt{w^t \Sigma w}} \underbrace{\|\Sigma^{-1/2} \mu\|}_{\sqrt{\mu^t \Sigma^{-1} \mu}}.$$

Therefore

$$\boxed{\mathcal{S}(w) \leq \sqrt{\mu^t \Sigma^{-1} \mu}}.$$

The bound applies to all nonzero positions, not just frontier portfolios.

When is the Sharpe-ratio bound attained?

For $\mu \neq 0$, equality in the upper bound requires positive alignment:

$$\Sigma^{1/2} w = k \Sigma^{-1/2} \mu, \qquad k > 0.$$

Equivalently,

$$\boxed{w = k \Sigma^{-1} \mu, \qquad k > 0}.$$

If $b = \mathbf{1}^t \Sigma^{-1} \mu > 0$, imposing $\mathbf{1}^t w = 1$ gives

$$w_T = \frac{\Sigma^{-1} \mu}{b}, \qquad \mathcal{S}(w_T) = \sqrt{\mu^t \Sigma^{-1} \mu}.$$

- The tangency portfolio attains the maximum Sharpe ratio.
- With a risk-free asset and no further constraints, risk aversion changes the scale of the optimal risky position, not its direction.

Practical implementation and limitations of Markowitz' approach

Markowitz in practice

The raw-return mean is $m = \mathbb{E}[R]$, while $\mu = m - r\mathbf{1}$ is the expected excess return. These parameters and Σ must be estimated from observations $R_1, \ldots, R_n \in \mathbb{R}^d$. For a Gaussian statistical model, suppose the observations are i.i.d. $\mathcal{N}(m, \Sigma)$ with $\Sigma \succ 0$. Their likelihood is

$$L_n(m, \Sigma) = (2\pi)^{-nd/2} (\det \Sigma)^{-n/2} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} (R_i - m)^t \Sigma^{-1} (R_i - m) \right).$$

Theorem

If the centred observations span $\mathbb{R}^d$, the maximum likelihood estimators are

$$\widehat{m} = \frac{1}{n} \sum_{i=1}^{n} R_i, \qquad \widehat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} (R_i - \widehat{m})(R_i - \widehat{m})^t.$$

The corresponding excess-return estimate is $\widehat{\mu} = \widehat{m} - r\mathbf{1}$.

Limitations of Markowitz portfolio theory

- **A criterion, not a distributional assumption.** Mean–variance optimisation does not require Gaussian returns. The Gaussian/exponential-utility justification is a special case.
- **Dispersion is not tail severity.** Standard deviation treats deviations above and below the mean symmetrically. A short out-of-the-money put can generate frequent small gains and rare large losses.
- **Estimation and inversion are delicate.** For the centred data matrix $X_c = [R_1 - \widehat{m}, \ldots, R_n - \widehat{m}] \in \mathbb{R}^{d\times n}$,

$$\widehat{\Sigma} = \frac{1}{n} X_c X_c^t, \qquad \operatorname{rank}(\widehat{\Sigma}) \leq \min(d, n-1).$$

Thus $n > d$ is necessary for invertibility; full row rank of X_c is also required. Even an invertible estimate can be ill-conditioned, leading to unstable portfolio weights.

Thank you!