Université Paris 1 Panthéon Sorbonne
Exam M2 IRFA - MMMEF, 2024-2025
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Exam: Market Risk Measures

16th December 2024

- Documents, calculator as well as cell phone are prohibited.
- The duration of the exam is **2h00**.
- It is forbidden to leave without returning the copy of your exam sheet.
- The exam is composed of **3 exercices** which can be treated independently. In an exercise, you can use the results from previous questions.

Name: **IRFA/MMMEF:**

Exercice 1 (Questions on the lectures (8 pts))

1. (2 pts) For a given loss L and confidence level $\alpha \in (0,1)$, recall the definitions of the Value-at-Risk at level α of L denoted by $\mathrm{VaR}_\alpha(L)$ and the Expected Shortfall, also known as Conditional Value-at-Risk, at level α of L denoted by $\mathrm{ES}_\alpha(L)$.
2. (1 pt) Show that the map $L \mapsto \mathrm{VaR}_\alpha(L)$ is non-decreasing, namely, if $L_1 \leqslant L_2$ *a.s.* then one has
$$\mathrm{VaR}_\alpha(L_1) \leqslant \mathrm{VaR}_\alpha(L_2).$$
3. (1 pt) Using the result of question 2, prove that $L \mapsto \mathrm{ES}_\alpha(L)$ is also a non-decreasing map.
4. (1 pt) What are the axioms/properties that a risk measure ρ should satisfy in order to be *coherent*? Explain in details each axiom.
5. (1 pt) Provide an example of a risk measure ρ which is not coherent. Explain in details why.
6. (2 pts) Let L be a loss random variable with exponential law with parameter $\lambda > 0$. Recall that its density is given by $f_L(x) = \lambda \exp(-\lambda x)$. Prove that
$$\mathrm{VaR}_\alpha(L) = \frac{-\ln(1-\alpha)}{\lambda}, \quad \text{and} \quad \mathrm{ES}_\alpha(L) = \frac{-\ln(1-\alpha)+1}{\lambda}.$$

Exercice 2 (Mesure de risque Expectile (6 pts)) Pour $\alpha \in (0,1)$, on définit la fonction $\ell_\alpha : \mathbb{R} \to \mathbb{R}_+$ par
$$\ell_\alpha(x) = \alpha(x_+)^2 + (1-\alpha)((-x)_+)^2 = \begin{cases} \alpha x^2 & \text{if } x \geqslant 0, \\ (1-\alpha)x^2 & \text{if } x \leqslant 0. \end{cases}$$
On rappelle la définition $x_+ = \max(0, x)$.

On considère une variable aléatoire réelle L représentant la perte d'un portefeuille sur un horizon donnée. On suppose que $L \in L^2(\mathbb{P})$ et que sa fonction de répartition est continue.

On définit une mesure de risque comme la solution du problème d'optimisation
$$\rho_\alpha(L) = \arg\min_{x \in \mathbb{R}} L_\alpha(x), \quad \text{où} \quad L_\alpha(x) := \mathbb{E}[\ell_\alpha(L-x)]. \tag{0.1}$$

1. (3 pts)
   (a) (1 pt) Montrer que la fonction L_α définie par (0.1) est convexe.
   (b) (1 pt) Montrer que L_α est continument différentiable sur $\mathbb{R}$ avec une dérivée donnée par
   $$L'_\alpha(x) = \mathbb{E}[-2\alpha(L-x)_+ + 2(1-\alpha)(x-L)_+], \quad x \in \mathbb{R}.$$
   (c) (1 pt) En utilisant le fait que ℓ_α est convexe et l'inégalité de Jensen, montrer que
   $$\lim_{x \to +\infty} L_\alpha(x) = \lim_{x \to -\infty} L_\alpha(x) = +\infty.$$
2. (2 pts) Déduire de la question 1. que L_α admet un minimum sur $\mathbb{R}$ et que $\rho_\alpha(L)$ est une solution de l'équation
$$\alpha \mathbb{E}[(L-x)_+] = (1-\alpha)\mathbb{E}[(x-L)_+]. \tag{0.2}$$

3. (2 pts) En étudiant les variations de la fonction $f : x \mapsto \alpha\mathbb{E}[(L-x)_+]-(1-\alpha)\mathbb{E}[(x-L)_+]$, montrer que (0.2) admet une unique solution.

4. (2 pts) Soit $\lambda > 0$. En utilisant la caractérisation précédente pour $\rho_\alpha(\lambda L)$ à l'aide de (0.2), prouver que $\rho_\alpha(\lambda L) = \lambda\rho_\alpha(L)$.
Quel est le nom de cette propriété dans le langage des mesures de risque?

5. (1 pt) Soit $d \in \mathbb{R}$. En utilisant la même méthode que la question précédente, montrer que

$$\rho_\alpha(L + d) = \rho_\alpha(L) + d.$$

Quel est le nom de cette propriété dans le langage des mesures de risque?

Exercice 3 (Characterization of the Expected Shortfall (6 pts))

Let L be a real-valued random variable with continuous cumulative distribution function and such that $L \in L^1(\mathbb{P})$.

1. (1 pt) For $\alpha \in (0, 1)$, we consider the following optimisation problem

$$\rho_\alpha(L) := \min_{\xi\in\mathbb{R}} \left\{ L_\alpha(\xi) := \xi + \frac{1}{1-\alpha}\mathbb{E}[(L-\xi)_+] \right\}. \tag{0.3}$$

Prove that the function L_α defined above is convex.

2. (1 pt) Using (0.3), prove that ρ_α is a sub-additive risk measure.

3. (1 pt) Prove that L_α is continuously differentiable on $\mathbb{R}$ and that its derivative satisfies:

$$L'_\alpha(\xi) = \frac{\mathbb{P}(L \leqslant \xi) - \alpha}{1-\alpha}, \quad \xi \in \mathbb{R}.$$

4. (1 pt) Prove that

$$\lim_{\xi\mapsto+\infty} L_\alpha(\xi) = \lim_{\xi\mapsto-\infty} L_\alpha(\xi) = +\infty.$$

Hint: Use Jensen's inequality.

5. (1 pt) Assuming that the cumulative distribution function of L is strictly increasing, deduce from the preceding question that the stochastic optimization problem of question 1 admits a unique solution $\xi^\star_\alpha$. Show that $\xi^\star_\alpha = \mathrm{VaR}_\alpha(L)$.

6. (1 pt) Conclude from the previous questions that

$$\min_{\xi\in\mathbb{R}} L_\alpha(\xi) = L_\alpha(\xi^\star_\alpha) = \mathrm{ES}_\alpha(L).$$