In fact, it is not only about "giving up". One could very well say that it is the quantity of good (2) per unit of good (1) that the consumer must "acquire" in order to "give up" a marginal quantity of good (1).

Yet, we generally say "give up". Why?

Because, *if the agent tries to obtain the maximum satisfaction* (we then say that he is **rational**) and *if he prefers more to less*, as is assumed in the model, then…

> the **MRS** of good (2) for good (1) is the *maximal* quantity of good (2) that the consumer is willing to give up in exchange for one unit of good (1). It is therefore his ***reserve price – in good (2) – for the purchase of good (1)***: the maximum price in good (2) that the agent is likely to agree to pay to acquire each unit of good (1), whatever the quantity of good (1) he acquires.

Indeed, whatever this quantity, including therefore a very small one, if the price of good (1) in good (2) is higher than the MRS, then the agent will not wish to buy good (1).

Knowing an agent's MRS at his basket of initial endowments is therefore knowing *his dispositions to exchange as a function of the posted prices*[1].

Suppose that the agent's MRS at his basket of initial endowments equals 4. This means that he is willing to give up at most 4 units of good (2) per unit of good (1) he acquires, otherwise his satisfaction will decrease. So if $p_1/p_2 = 5$, then good (1) is too expensive for him. He will not wish to buy any. On the other hand, he is likely to buy some if $p_1/p_2 = 3$.

### 3. Utility functions (reminders)

#### a. an ordinal function

As the preference relation is a complete preorder, it can be *represented by a* function, which by habit we call a *utility function* and which is generally denoted $u(\cdot)$. This function from $(\mathbb{R}_+)^n$ to $\mathbb{R}_+$ respects the consumer's preferences. In other words, to each basket of goods $Q$, it associates a real number $u(Q)$ such that:

$$Q \succsim Q' \Leftrightarrow u(Q) \geq u(Q')$$
$$Q \sim Q' \Leftrightarrow u(Q) = u(Q').$$

[1] **Warning**: just as the price of a good in euros is the quantity of euros that must be given up to acquire one unit of that good, the price of good (1) in good (2) – denoted $p_1/p_2$ – is the quantity of good (2) that must be given up to acquire one unit of good (1).

*Example*

Consider an individual whose preferences obey the usual assumptions (completeness, reflexivity, transitivity) and who, faced with the baskets of goods $Q_1 = (2, 8)$, $Q_2 = (4, 2)$, $Q_3 = (3, 3)$, says he has the following preferences:

$$Q_1 \succ Q_3,\ Q_1 \sim Q_2.$$

Can these preferences be represented by the following utility functions:

$$u_1(q_1, q_2) = q_1^2 q_2,\ u_2(q_1, q_2) = q_1 \times \sqrt{q_2},\ u_3(q_1, q_2) = 2lnq_1 + lnq_2,\ u_4(q_1, q_2) = 2q_1 + q_2\ ?$$

Since we have: $u_1(Q_1) = 32$, $u_1(Q_2) = 32$, $u_1(Q_3) = 27$, the function $u_1(.)$ represents this consumer's preferences, since we have:

$$u_1(Q_1) = u_1(Q_2) > u_1(Q_3).$$

The same goes for $u_2(.)$ and $u_3(.)$. Indeed, in the perfect competition model, the approach is **ordinal** and not **cardinal**[2]: what matters is not that $u_1(Q_1) = 32$, but that $u_1(Q_1) = u_1(Q_2) > u_1(Q_3)$; an infinity of functions can therefore do. In fact, if we find one, we know that all the strictly increasing transformations of this function will do as well.

Indeed, $f(.)$ is a **strictly increasing function** if and only if,

for any pair $(x, x')$ such that $x > x'$,

we have: $f(x) > f(x')$.

It is therefore a function that *preserves order*.

Now:

. $u_2(\cdot) = f[u_1(\cdot)]$ with $f(x) = \sqrt{x}$ (and the square root function is increasing) and

. $u_3(\cdot) = f[u_1(\cdot)]$ with $f(x) = \ln x$ (and the natural logarithm function is increasing).

On the other hand, the function $u_4(\cdot)$ does not represent this consumer's preferences. Indeed, we have: $u_4(Q_1) = 12 > u_4(Q_2) = 10$ which is incompatible with $Q_1 \sim Q_2$.

In tutorials, we will work a lot with Cobb-Douglas type utility functions. See the box below THE COBB-DOUGLAS TYPE FUNCTION.

THE COBB-DOUGLAS TYPE FUNCTION

**The Cobb-Douglas function** takes its name from Charles W. Cobb (mathematician) and Paul H. Douglas (economist). It has the form:

$$u(q_1, q_2) = q_1^{\alpha} q_2^{\beta},\ \alpha > 0\ and\ \beta > 0.$$

If an agent's preferences can be represented by such a function $u(\cdot)$ defined by:

$$u(q_1, q_2) = q_1^{\alpha} q_2^{\beta}, \quad with\ \alpha > 0\ and\ \beta > 0,$$

then, this function being ordinal in this model, these preferences can also be represented (among others) by the functions:

[2]You will come back to this question in tutorials.

- $v(\cdot)$ defined by $v(q_1, q_2) = [u(q_1, q_2)]^\gamma$ with $\gamma > 0$, which gives: $v(q_1, q_2) = q_1^{\alpha\gamma} q_2^{\beta\gamma}$.
- and $w(\cdot)$ defined on $(\mathbb{R}_+^*)^2$ by $w(q_1, q_2) = \ln[u(q_1, q_2)] = \alpha \ln q_1 + \beta \ln q_2$. This last function is called **log-linear** because it is a linear combination of the logarithms of the variables. The function $u_3(\cdot)$ of the previous example is of this type.

### b. Utility function and indifference curves

Reminder from L1. When an agent's preferences are represented by the utility function u(·), the indifference curve passing through basket $Q_0$ links all the baskets of goods $(q_1 , q_2)$ giving the agent the same satisfaction as basket $Q_0$. Its equation is therefore: $u(q_1 , q_2) = u(Q_0)$.

*Example*

Consider an agent A whose preferences can be represented by the function $u_A(\cdot)$ defined by:

$$u_A(q_1, q_2) = q_1^2 q_2 .$$

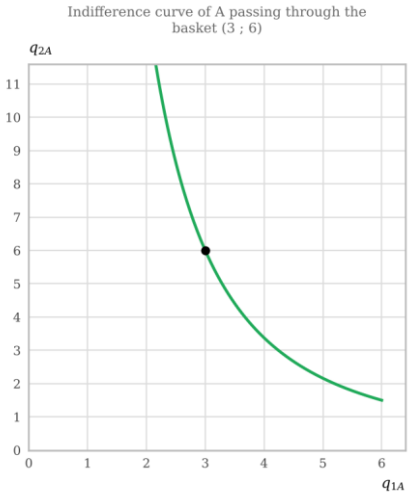
If A's initial endowments are: $Q_{0A} = (3 , 6)$. A's indifference curve passing through this basket has equation: $u_A(q_1 , q_2) = u_A(3 , 6) = 3^2 \times 6 = 54$, which gives:

$$q_1^2 q_2 = 54,$$

or again:

$$q_2 = \frac{54}{q_1^2}$$

See graph at right.




When utility functions are of the *Cobb-Douglas* or *log-linear* type, the indifference curves are always continuous, decreasing, convex and asymptotic to the axes. See the box below: INDIFFERENCE CURVES OF COBB-DOUGLAS TYPE FUNCTIONS.

INDIFFERENCE CURVES OF COBB-DOUGLAS TYPE FUNCTIONS

When an agent's preferences can be represented by a *Cobb-Douglas type utility function*, the *indifference curves* therefore have an equation of the form:

$$q_1^\alpha q_2^\beta = c$$

where $\alpha > 0$, $\beta > 0$ and $c$ is a strictly positive constant for $q_1 > 0$ and $q_2 > 0$[3]. Now:

$$q_1^\alpha q_2^\beta = c \Rightarrow q_2^\beta = \frac{c}{q_1^\alpha} \Rightarrow q_2 = \left(\frac{c}{q_1^\alpha}\right)^{1/\beta} = \frac{c^{1/\beta}}{q_1^{\alpha/\beta}} = k \times q_1^{-\alpha/\beta}$$

where $k = c^{1/\beta}$ is a positive constant.

The indifference curves therefore have an equation of the form:

[3] If $q_1 = 0$ or $q_2 = 0$ then $q_1^\alpha q_2^\beta = 0$. All baskets of the form $(q_1 , 0)$ or $(0 , q_2)$ are therefore located on the same indifference curve, which therefore coincides with the axes (with the positive part of the axes, more precisely). We therefore examine here the case where $q_1 > 0$ and $q_2 > 0$.

$$q_2 = k \times q_1^{-\alpha/\beta}$$

And the indifference curves of the log-linear function $w(\cdot)$ defined by $w(q_1, q_2) = \alpha \ln q_1 + \beta \ln q_2$ obviously have an equation of the same form.

These indifference curves are continuous (power function), *decreasing*, *convex* and *asymptotic to the axes*:

- They are decreasing: the function $g(\cdot)$ defined by $g(q_1) = k \times q_1^{-\alpha/\beta}$ is decreasing on $\mathbb{R}_+^*$ because its first derivative is negative on this set. Indeed:

$$g'(q_1) = k \times \frac{-\alpha}{\beta} q_1^{-\frac{\alpha}{\beta}-1} = -\frac{k\alpha}{\beta} q_1^{-\frac{\alpha+\beta}{\beta}} < 0 \; since \; k > 0, \alpha > 0 \; and \; \beta > 0.$$

- They are convex: the function $g(\cdot)$ defined by $g(q_1) = k \times q_1^{-\alpha/\beta}$ is strictly convex on $\mathbb{R}_+^*$ because its second derivative is positive on this set. Indeed:

$$g''(q_1) = -\frac{k\alpha}{\beta}\left(-\frac{\alpha+\beta}{\beta}\right) q_1^{-\frac{\alpha}{\beta}-2} = \frac{k\alpha(\alpha+\beta)}{\beta^2} q_1^{-\frac{\alpha}{\beta}-2} > 0 \; since \; k > 0, \alpha > 0 \; and \; \beta > 0$$

- They are asymptotic to the axes: indeed, since $k > 0$, $\alpha > 0$ and $\beta > 0$, we have:

$$g(q_1) = k \times q_1^{-\alpha/\beta} \underset{q_1 \to 0^+}{\longrightarrow} +\infty \quad and \quad g(q_1) = k \times q_1^{-\alpha/\beta} \underset{q_1 \to +\infty}{\longrightarrow} 0^+.$$

### c. Link between utility function and marginal rate of substitution

Reminder from L1. If an agent's preferences can be represented by the utility function $u(\cdot)$, then we have:

$$MRS_{2/1}(q_1, q_2) = \frac{u'_{q_1}(q_1, q_2)}{u'_{q_2}(q_1, q_2)}.$$

The MRS at $Q_0$ is therefore equal to the ratio of the partial derivatives of the utility function at $Q_0$.

In a cardinal approach, one would say that it is the ratio of marginal utilities. The approach being ordinal here, the concept of marginal utility however makes no sense.

In the case of Cobb-Douglas type utility functions, the marginal rate of substitution has a very specific form. When the utility function $u(\cdot)$ is defined by $u(q_1, q_2) = q_1^{\alpha} q_2^{\beta}$, with $\alpha > 0$ and $\beta > 0$, we indeed have:

$$MRS_{2/1}(q_1, q_2) = \frac{u'_{q_1}(q_1, q_2)}{u'_{q_2}(q_1, q_2)} = \frac{\alpha q_1^{\alpha-1} q_2^{\beta}}{\beta q_1^{\alpha} q_2^{\beta-1}} = \frac{\alpha q_2^{\beta} q_2^{1-\beta}}{\beta q_1^{\alpha} q_1^{1-\alpha}} = \frac{\alpha q_2^{\beta+1-\beta}}{\beta q_1^{\alpha+1-\alpha}} = \frac{\alpha q_2}{\beta q_1}.$$

MRS IN THE CASE OF A COBB-DOUGLAS TYPE FUNCTION

When the utility function $u(\cdot)$ is defined by $u(q_1, q_2) = q_1^{\alpha} q_2^{\beta}$, with $\alpha > 0$ and $\beta > 0$, we have:

$$MRS_{2/1}(q_1, q_2) = \frac{\alpha q_2}{\beta q_1}.$$

And we obviously get exactly the same MRS for the log-linear function $w(\cdot)$ defined by $w(q_1, q_2) = \alpha \ln q_1 + \beta \ln q_2$.

In the case of the example of agent A seen above, where $\alpha = 2$ and $\beta = 1$, we have:

$$MRS_{2/1}(q_1, q_2) = \frac{2q_2}{q_1}.$$

Had we had α = 4 and β = 2, the MRS would have been the same, since:

$$MRS_{2/1}(q_1, q_2) = \frac{4q_2}{2q_1} = \frac{2q_2}{q_1}.$$

This is perfectly normal. Indeed, utility functions being ordinal, the function defined by $u(q_1\ ,\ q_2) = q_1^4 q_2^2$ represents the same preferences as the function $u_A(q_1\ ,\ q_2) = q_1^2 q_2$ of which it is an increasing transformation.

## II. The exchange problem

The perfect competition model is fundamentally an exchange model. The question of exchange itself divides into two (related) questions: that of the prices at which goods are exchanged (the exchange rate) and that of each one's gain from exchange, in other words the sharing of the gain from exchange.

We will reason in the simplest case of an economy with two agents A and B and two goods (1) and (2).

Each agent is characterized by his tastes (his preferences, his indifference map) and what he owns (his initial endowments).

- A's initial endowments are: $(q_{10A}\ ,\ q_{20A}) = Q_{0A}$; B's are: $(q_{10B}\ ,\ q_{20B}) = Q_{0B}$.
- The indifference curves are assumed to have the "usual" shape, *i.e.* to be of hyperbolic type.

To "make things concrete" a little, we will give values to these initial endowments and define the utility functions, but it is exactly the same reasoning for any initial endowments and utility functions such that the indifference curves are continuous, decreasing, convex and asymptotic to the axes.

We will therefore take up again the example of agent A encountered above: his initial endowments are: (3 , 6) and his preferences can be represented by the function $u_A(q_1, q_2) = q_1^2 q_2$. A's indifference curve passing through (3 , 6) therefore has equation: $q_2 = 54/q_1^2$ (see above).

Suppose that B's initial endowments are: (5 , 5) and that $u_B(q_1, q_2) = q_1\ q_2^4$. B's indifference curve passing through basket (5 , 5) therefore has equation:

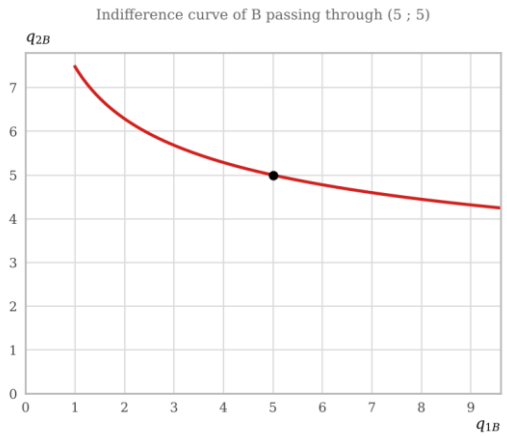
$$u_B(q_1, q_2) = u_B(5\ ;\ 5),$$

which gives:

$$q_1\ q_2^4 = 3125$$

namely:

$$q_2 = \frac{5^{5/4}}{q_1^{1/4}}$$ (see graph at right).




*Remark*: MRS(5 ; 5) = ¼.

We will now suppose that, in this economy, there is *no production*: it is therefore a *pure exchange economy*.

The quantity of good (1) available in the economy, denoted $\bar{q}_1$, is therefore the sum of the two agents' initial endowments in good (1):

$$\bar{q}_1 = q_{10A} + q_{10B} = 3 + 5 = 8.$$

Same for $\bar{q}_2$, the quantity of good (2) available in the economy. It is the sum of the two agents' initial endowments in good (2):

$$\bar{q}_2 = q_{20A} + q_{20B} = 6 + 5 = 11.$$

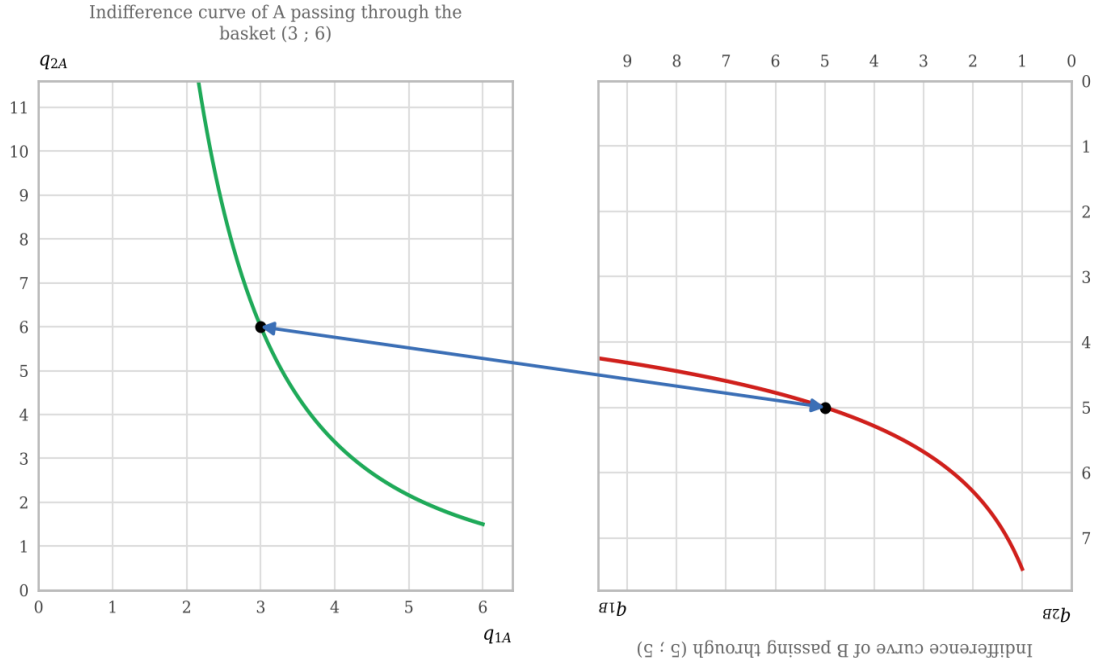
In neoclassical theory, we are interested in the satisfaction that the consumer derives from the consumption of goods and we suppose that the consumer is **rational**, in the particular sense where *he tries to obtain the greatest possible satisfaction*. We are therefore interested here in the way A and B coordinate to improve their satisfaction.

At this stage, everything is possible. To improve his situation, A can kill – or steal from – B and consume all the goods available in the economy. This case is ruled out by neoclassical theory. It is supposed, indeed, that a certain number of social rules exist. One could imagine that everything is pooled and shared out according to individuals' needs. These are not the rules retained in the model. It is supposed there that *A and B respect the private property of endowments and can improve their satisfaction only through voluntary exchange*. It is therefore a market society.

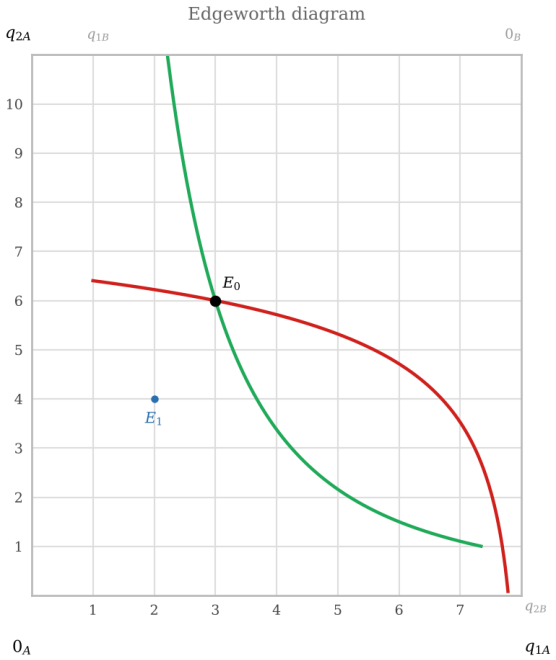
## Graphical analysis

To represent this situation, and the possible exchanges, a diagram called an **Edgeworth diagram** or **Edgeworth box** is used.

Graphically, an Edgeworth box is built by *flipping* B's indifference curve and *superimposing* it on A's *so that the two baskets of initial endowments $Q_{0A}$ = (3 ; 4) and $Q_{0B}$ = (5 ; 5) are superimposed.*




This gives the following rectangle where **$Q_{0A}$ and $Q_{0B}$ are one and the same point renamed $Q_0$**. It is this rectangle that is called an Edgeworth *box* – or *diagram*.




The base – or length – of this rectangle represents the quantity of good (1) available in the economy, $\bar{q}_1$, namely 8. Its height – or width – represents the quantity of good (2) available in the economy, $\bar{q}_2$, namely 11. **Check this carefully!**

Each point of this diagram is a possible allocation of the economy's resources between A and B. Such an allocation is called a **feasible state** of the economy.

Consider an economy with two agents, A and B, and two goods, (1) and (2), available in quantities $\bar{q}_1$ and $\bar{q}_2$ respectively.

A **feasible state** of this economy is any set $E$:

$$E = \{(q_{1A}, q_{2A}), (q_{1B}, q_{2B})\} \; with \; q_{1A} + q_{1B} = \bar{q}_1 \; and \; q_{2A} + q_{2B} = \bar{q}_2,$$

where $(q_{1A}, q_{2A})$ is the basket of goods owned by A and $(q_{1B}, q_{2B})$ the basket of goods owned by B. It is therefore *an allocation of the available resources between the two agents A and B*.

In our example, a feasible state is:

$$E = \{(q_{1A}, q_{2A}), (q_{1B}, q_{2B})\} \; with \; q_{1A} + q_{1B} = 8 \; and \; q_{2A} + q_{2B} = 11.$$

Thus, for example, the initial allocation $Q_0 = \{Q_{0A}, Q_{0B}\} = \{(3 ; 6) , (5 ; 5)\}$ is a feasible state of this economy (and for good reason), which appears as the point (3 ; 6) when looking at the Edgeworth diagram right side up, and as the basket (5 ; 5) when looking at it upside down.

Likewise, the feasible state $E_1 = \{(2 ; 4) , (6 ; 7)\}$, the blue point on the graph above, appears as the point (2 ; 4) when looking at the said graph right side up (A then has two goods (1) and four goods (2)) and as the point (6 ; 7) when looking at it upside down (B then has six goods (1) and seven goods (2)). And we do have (8 ; 11) = (2 ; 4) + (6 ; 7): together, A and B do own all the resources available in the economy.

The set of feasible states that A prefers to his basket of initial endowments, $Q_{0A}$ = (3 ; 6), is located above his indifference curve passing through $Q_{0A}$. These are the states of the zone hatched in green in the graph below.

The set of feasible states that B prefers to his basket of initial endowments, $Q_{0B}$ = (5 ; 5), is located above his indifference curve passing through $Q_{0B}$. But, this curve being upside down in the Edgeworth diagram, this zone is located below this indifference curve. These are therefore the feasible states of the zone hatched in red in the graph below.




How will A and B coordinate? What exchange rate will result?

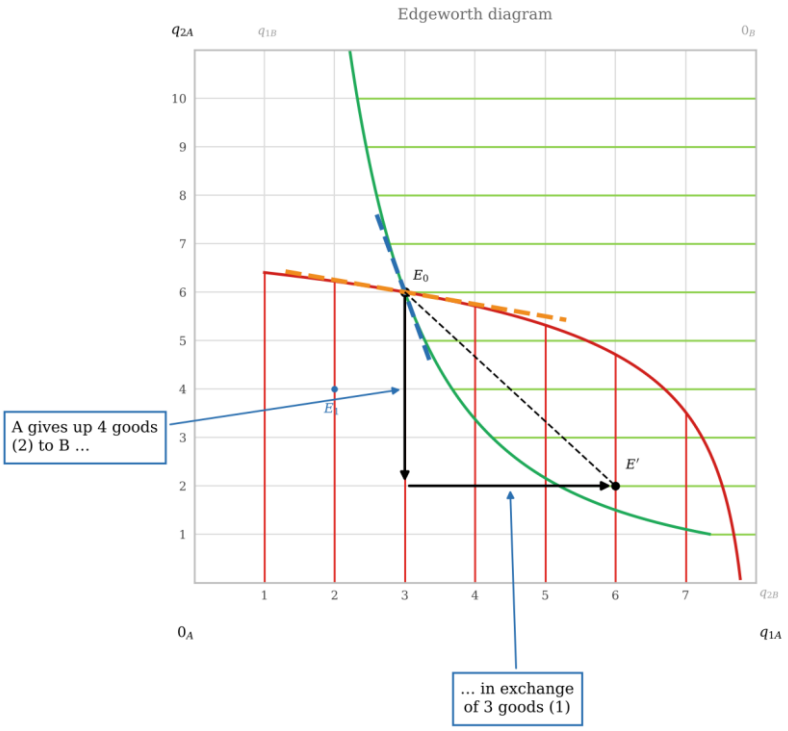
*If exchanges are voluntary*, the set of feasible states that A and B are both likely to find acceptable are those located in the zone hatched in both green and red, called the *lens*, because of its shape. For moving from feasible state $E_0$ to any feasible state in the lens benefits both.

How will A and B coordinate? What exchange rate will result?

*If exchanges are voluntary*, the set of feasible states that A and B are both likely to find acceptable are those located in the zone hatched in both green and red, called the *lens*, because of its shape. For moving from feasible state $Q_0$ to any feasible state in the lens benefits both.

Example: suppose that A gives up 4 units of good (2) to B in exchange for 3 units of good (1) (the exchange rate – in other words the price of good (1) in good (2) – is then equal to 4/3; this exchange rate is represented by the absolute value of the slope of segment [$E_0E'$] on the graph below):

- A then ends up with the basket $Q'_A = Q_{0A} + (+3 ; -4) = (3 ; 6) + (3 ; -4) = (6 ; 2)$
- and B, with $Q'_B = Q_{0B} + (-3 ; +4) = (5 ; 5) + (-3 ; 4) = (2 ; 9)$.
- On the graph below, they have therefore moved from feasible state $E_0 = \{(3 ; 6) , (5 ; 5)\}$ to feasible state $E' = \{(6 ; 2) , (2 ; 9)\}$, which is in the lens.




Admittedly, they both prefer feasible state E' to initial state $E_0$ (since E' is in the lens), but there is no reason why they would agree on this allocation of resources rather than another. Who proposes this exchange? Why would they accept it? If it is B who proposes this exchange, won't A find that he must give a lot of good (2) for too little good (1)? That he could have given up three, or even two, and yet improved B's satisfaction? And then is it "rational" in the sense defined above, where it is a question of the greatest satisfaction they can obtain?

In short, if no assumption is added, one cannot say what quantities A and B will exchange, nor *a fortiori* at what exchange rate or on which allocation they will agree, if indeed they agree at all.