Université Paris 1 Panthéon Sorbonne
Exam M2 MMMEF, 2022-2023

Exam: Calibration in quantitative finance

14th March 2023

- Documents and cell phone are prohibited.
- The duration of the exam is **2h00**.
- It is forbidden to leave without returning the copy of your exam sheet.
- The exam is composed of **2 exercices** which can be treated independently. In an exercise, you can use the results from the previous questions.

Exercice 1 (Questions on the lectures **(7 pts)**)

1. (2 pts) Prove the two following non-arbitrage bounds on the price $C(t,T,S,K)$ of a Call option with maturity T and strike K evaluated at time t when the stock price is S:

$$(S - Ke^{-r(T-t)})_+ \leqslant C(t,T,S,K) \leqslant S.$$

2. (3 pts) Suppose that we observe on the market several Call prices $C(T,K)$ for different strikes K and maturities T.
   (a) What is the definition of the implied volatility $\tilde{I}(T,K)$ for the characteristics (T,K)?
   (b) Under which assumption on $C(T,K)$ is it well defined?
   (c) Can you provide a simple numerical method to compute it?
3. (1 pt) What is definition of the skew phenomenon?
4. (1 pt) What is definition of the smile phenomenon?

Exercice 2 (15 pts)

Part 1: Static hedge of European options (8.5 pts)

We consider a stock which price at time t is given S_t taking values in $\mathbb{R}_+$. We assume that the process $(S_t)_{t\geqslant 0}$ admits a positive transition density given by $p(t,x,T,y)$, namely, one has

$$\mathbb{E}[h(S_T)|S_t = x] = \int_{\mathbb{R}_+} h(y)\, p(t,x,T,y)\, dy, \quad 0 \leqslant t \leqslant T,$$

for any measurable map $h : \mathbb{R}_+ \to \mathbb{R}$ with at most linear growth. We denote by $C(t,T,S,K)$ and $P(t,T,S,K)$ the prices of a Call and Put option evaluated at time t when the stock price is given by S with maturity T and strike K. We will denote by r the interest rate which is assumed to be constant. We also introduce the *forward price of the stock* at time t defined by

$$F_t = S_t \exp(r(T-t)). \tag{0.1}$$

1. (1.5 pts) Recall the Call-Put parity relation linking $C(t,T,S_t,K)$ and $P(t,T,S_t,K)$. Its proof is not required. Deduce that

$$\partial_K P(t,T,S,K) - \partial_K C(t,T,S,K) = -\exp(-r(T-t)).$$

2. (1 pt) For which specific value of K do we have that $C(t,T,S_t,K) - P(t,T,S_t,K) = 0$?
3. (2 pts) Prove the two following identities

$$p(t,S,T,K) = \exp(r(T-t))\frac{\partial^2}{\partial K^2}C(t,T,S,K) = \exp(r(T-t))\frac{\partial^2}{\partial K^2}P(t,T,S,K).$$

4. (1 pt) Prove the following relation: for any $F \geqslant 0$

$$e^{-r(T-t)}\mathbb{E}[h(S_T)|\, S_t = s] = \int_0^F h(K)\frac{\partial^2}{\partial K^2}P(t,T,S,K)\, dK + \int_F^\infty h(K)\frac{\partial^2}{\partial K^2}C(t,T,S,K)\, dK.$$

5. (2 pts) Deduce from the results of questions 1, 2 and 4 that

$$e^{-r(T-t)}\mathbb{E}[h(S_T)|\ S_t] = e^{-r(T-t)}h(F_t) + \int_0^{F_t} h''(K)P(t,T,S_t,K)\,dK \\ + \int_{F_t}^{\infty} h''(K)C(t,T,S_t,K)\,dK, \tag{0.2}$$

recalling that F_t is given by (0.1).

6. (1 pt) What is the financial interpretation of the previous identity?

Part 2: Example: amortizing options (4.5 pts)

A common variation on the payoff of the standard European Call option which is particularly attractive in high volatility periods of the stock price is given by the *amortizing Call option* with strike L with payoff

$$h(S_T) = \frac{(S_T - L)_+}{S_T}. \tag{0.3}$$

The aim of this section is to establish the corresponding identity (0.2) for this particular class of options. Since the function $x \mapsto x_+$ is not twice continuously differentiable in the classical sense, it is clear that (0.3) is not $\mathcal{C}^2(\mathbb{R}_+\backslash\{0\})$ so that we cannot apply directly (0.2). We thus proceed by an approximation argument. We let

$$f_\varepsilon(x) = \frac{(x+\frac{\varepsilon}{2})^2}{2\varepsilon}\mathbf{1}_{x\in[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]} + x\mathbf{1}_{x>\frac{\varepsilon}{2}}, \quad \varepsilon > 0.$$

1. (1.5 pts) Prove that $f_\varepsilon \to x_+$ for any $x \in \mathbb{R}_+$ and compute f'_ε and f''_ε for any $\varepsilon > 0$.
2. Using the previous approximation of the positive part, we let

$$h_\varepsilon(s) = \frac{f_\varepsilon(s-L)}{s}.$$

In order to simplify, **we assume that** $L > F_t$.

(a) (2 pts) Prove that

$$\lim_{\varepsilon\downarrow 0}\int_0^{F_t} h''_\varepsilon(K)P(t,T,S_t,K)\,dK = 0$$

(b) (2 pts) By applying (0.2) to h_ε instead of h and passing to the limit as $\varepsilon \downarrow 0$, prove that

$$e^{-r(T-t)}\mathbb{E}[h(S_T)|\ S_t] = e^{-r(T-t)}\frac{C(t,T,S_t,L)}{L} - 2L\int_L^{\infty}\frac{C(t,T,S_t,K)}{K^3}\,dK.$$

3. (1 pt) What is the financial interpretation of the previous identity?