Portfolio choice theory and asset pricing

Tutorials : Risk measures

Exercise 1

1. For $X \sim \mathcal{N}(0,1)$, compute $\mathbb{E}(X^4)$.

2. Prove that a quadratic utility function satisfies the *Mean-Variance Assumption*.

3. **a.** For a general utility function U, recall why the absolute risk aversion is measured by $-\frac{U''}{U'}$ (Arrow-Prat measure of ARA).

b. Prove that quadratic utility implies an absolute risk aversion increasing with wealth.

Exercise 2. We assume that all assets (or portfolios) returns are Gaussian and we denote by U the utility function of the investor (U is then positively monotone and concave). We denote by W the final value of the portfolio chosen by the investor and by R the (rate of) return of this portfolio.

1. Prove that $\mathbb{E}(U(W))$ can be written : $f(\mathbb{E}(R), \sigma(R))$ with $f(x,y) = E(U(W_0(1+x+yX)))$ where $X \sim \mathcal{N}(0,1)$.

2. Deduce that the investor has preference for expected return and aversion for return standard deviation, i.e. the *Mean-Variance Assumption* is satisfied (as aversion for return standard deviation implies aversion for return variance).

Exercise 3. Reminder on correlation and on Gaussian variables.

1. Prove that the correlation between two variables is not changed by centering and reducing one or two of these variables (i.e. e.g. replacing X with $\frac{X-\mathbb{E}(X)}{\sqrt{V(X)}}$).

2. **a.** Prove that $Cov(X,Y)^2 \leq V(X)V(Y)$ for any random variables X, Y (this is a known property, you will re-prove it by studying the variance of $X - \frac{Cov(X,Y)}{V(Y)}Y$ if Y is not constant).

b. When do we have equality ?

3. Uncorrelated variables are not independent in general. Check it for example on X and X^2 for a good choice of X.

4. Is a linear combination of real Gaussian variables a Gaussian variable ?

Exercise 4.

1. Do we still have $R = \sum_i x_i R_i$ if x_i is the quantity of asset i in the portfolio page 4 (instead of being the proportion of wealth invested in asset i) ?

2. R_i and Σ defined in the notes. Let A be a matrix $N \times N$. We define N random variables $W_1, ..., W_N$ by : $\begin{pmatrix} W_1 \\ ... \\ W_N \end{pmatrix} = A \begin{pmatrix} R_1 \\ ... \\ R_N \end{pmatrix}$.

Prove that the covariance matrix of the variables $W_1, ..., W_N$ is $A\Sigma\, {}^tA$.

3. Prove by direct computation that the gradient of $X \mapsto {}^tXAX$ is $(A + {}^tA)X$.

Exercise 5. Prove that in the case of two correlated assets, the variance of the portfolio return is minimized for $x^* = \dfrac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$.

Exercise 6. Case of 2 assets, with short sales allowed (the computations in the case "short sales not allowed" were done during the class). We assume $\sigma_1 \leq \sigma_2$.

1. Check that in that case, the minimum variance frontier coincides with the opportunity set.

2. Study the case $\rho = 1$: in the standard deviation/expected return plane, describe the opportunity set, the GMV portfolio(s), and the efficient frontier.

$|\rho| = 1$ is excluded in the next questions.

3. Check that the GMV portfolio we obtained in the general case (for N assets) matches, for $N = 2$, what we directly computed in that case.

4. Compute the return variance of the GMV portfolio.

5. What are the possible values of $V(R_{GMV})$ when ρ changes ?

In particular, we drew page 9 the minimum variance frontier in the plane return standard deviation/expected return. Two GMV portfolios are shown on the first figure page 8, depending on the relative value of ρ and $\frac{\sigma_1}{\sigma_2}$. Can the two corresponding minimum variances be compared ?

6. In the N risky assets framework, we proved that the covariance of the GMV portfolio return with the return of any combination of assets (not only a minimum variance portfolio) is : $Cov(R_{GMV}, R_X) = \frac{1}{c} = V(R_{GMV})$.

Give another proof, by contradiction, considering combinations of GMV and another asset.

Exercise 7. (after the paragraph 2.4.2) Draw the efficient frontier

a. if investors cannot borrow at all,

b. if investors can lend at one rate r, but must pay a higher rate r' to borrow.

Exercise 8. Prove that the minimum variance frontier is an hyperbola when expected return is plotted against return standard deviation (ie volatility).

Exercise 9. Prove that the line $(GMV\ X)$ in the variance/mean plane cuts the expected return (ordinate) axis at point $(0, \mu_{X^\perp})$.

Exercise 10. Risky assets only. Let X be a minimum variance portfolio.
Recall the covariance between R_X and R_Y for any portfolio Y (computed page 15 in the course).
From the result :
- derive the equation of the minimum variance frontier as a parabola.
- compute the covariance of the GMV portfolio return with the return of any combination of assets.

Exercise 11. Compute the global minimum variance portfolio directly by minimising the variance.

Exercise 12. We consider N risky assets.
R_i denotes the return of the asset i for $i = 1, ..., N$, and we take $E_R = \begin{pmatrix} E(R_1) \\ ... \\ E(R_N) \end{pmatrix}$.

For $i, j = 1, ..., N$, we denote by σ_{ij} the covariance between R_i and R_j, and by Σ the returns covariance matrix for the N assets, assumed invertible.

Let $\sigma > 0$ and $\mathcal{L} = {}^tE_R X + \lambda\,(\sigma^2 - {}^tX\Sigma X) + \gamma\,(1 - {}^t\mathbf{1}X)$ for some constants λ and γ.

1. For which optimisation problem is this the Lagrangian ?

2. Prove that when X is a solution of this problem, we have : $X = \dfrac{1}{2\lambda}\Sigma^{-1}(E_R - \gamma\mathbf{1})$.

3. Prove that λ satisfies $(2\lambda\sigma)^2 = b - 2\gamma a + \gamma^2 c$, with a, b and c constants to be specified, and compute γc as a function of λ.

4. Deduce that $(2\lambda)^2(\sigma^2 c - 1) = d$, where you will define d.

5. Prove, under an additional assumption, that $X = \pm\sqrt{\dfrac{c\sigma^2 - 1}{d}}\Sigma^{-1}(E_R - \gamma\mathbf{1})$.

6. Study the set of the solution X when σ changes in the $(V(\cdot), E(\cdot))$ plane, by computing its equation. Which point of this set has the minimum variance ?

In the remaining exercises, there exists a risk-free asset.

Exercise 13. Compute the minimum variance frontier when there is a risk-free asset.

Exercise 14. Prove that when $r \neq \frac{a}{c}$, $\mu_T - \frac{a}{c}$ has same sign as $\frac{a}{c} - r$.

Exercise 15. Portfolio optimisation when there exists a risk-free asset : study the case $r = \frac{a}{c}$:

1. Prove that the 2 lines constituting the MV frontier correspond to the asymptotes of the hyperbola.
2. Prove that investing in A costs nothing.
3. Prove that the portfolio $\begin{pmatrix} 1 \\ X_A \end{pmatrix}$ is efficient and compute its expected return and its return variance.
4. Prove that the efficient frontier is $\left\{ \alpha_X \text{ portf} \begin{pmatrix} 1 \\ X_A \end{pmatrix} + (1 - \alpha_X) \text{risk-free asset} \mid \alpha_X \geq 0 \right\}$.

Exercise 16. N risky assets and there exists a risk-free asset.

1. The Sharpe ratio of a portfolio Y (built with the $N+1$ assets) is defined as $Sh(Y) = \dfrac{\mathbb{E}(R_Y) - r}{\sqrt{V(R_Y)}}$. Interpret that quantity.
2. We consider a portfolio of risky assets X with $Sh(X) > 0$. Prove that if Y is the combination of a positive investment in X and of some risk-free asset, then $Sh(Y) = Sh(X)$.
3. Recall the equation for the efficient frontier and draw it in the (return standard deviation, expected return) plane.
4. We consider the case where the tangent portfolio is efficient. Deduce from the figure of question **3** (its slope is denoted by f) the maximum possible value of the Sharpe ratio. For which portfolio of risky assets is this maximum attained ?
5. Prove the result of question **4.** by direct computation.
   (Hint : the condition $\frac{\partial \mathcal{L}}{\partial X} = 0$ gives an equation (E). You will pre-multiply (E) by ${}^tX$.)

Exercise 17. Portfolio optimisation when there exists a risk-free asset, and the risky assets' returns are all independent. We assume that the condition ensuring the existence of the tangent portfolio is satisfied.

1. Compute the tangent portfolio.
2. Check which weights in X_T are positive, negative.
   Prove that an efficient portfolio is long in asset i iff $\mathbb{E}(R_i) > r$.
   What is the highest weight ?
3. Compute the MV portfolio with expected return equal to μ.